π AND $\frac{22}{7}$ with IT'S COUSINS AND ORIGIN

SOUGATA PANDA

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Abstract

Here at first I have presented a simple problem related to π and then I have presented one generalised version of this, and at last presented some ideas and some interesting series related to π

1 A simple proof related to pi:

The first published statement of this result was in 1971 by Dalzell. It was also presented as a problem in 1968 *Putnam Competition* and later it also came in *JEE-Adv*. The problem was to compute the integral,

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$$

I encourage readers to try it at first.
Sol:

$$\begin{aligned}\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx &= \int_0^1 \left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right)\,dx \\ &= \left[\frac{x^7}{7}-\frac{2x^6}{3}+x^5-\frac{4x^3}{3}+4x\right]_0^1 - 4\left[\tan^{-1}(x)\right]_0^1 \\ &= \left[\frac{x^7}{7}-\frac{2x^6}{3}+x^5-\frac{4x^3}{3}+4x\right]_0^1 - \pi \\ &= \frac{1}{7}-\frac{2}{3}+1-\frac{4}{3}+4-\pi \\ &= \frac{22}{7}-\pi\end{aligned}$$

Now observe one thing,

$$\frac{x^4(1-x)^4}{1+x^2} > 0 \implies \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx > 0 \text{ as } 1 > 0$$

Hence,

$$\boldsymbol{\frac{22}{7} > \pi}$$

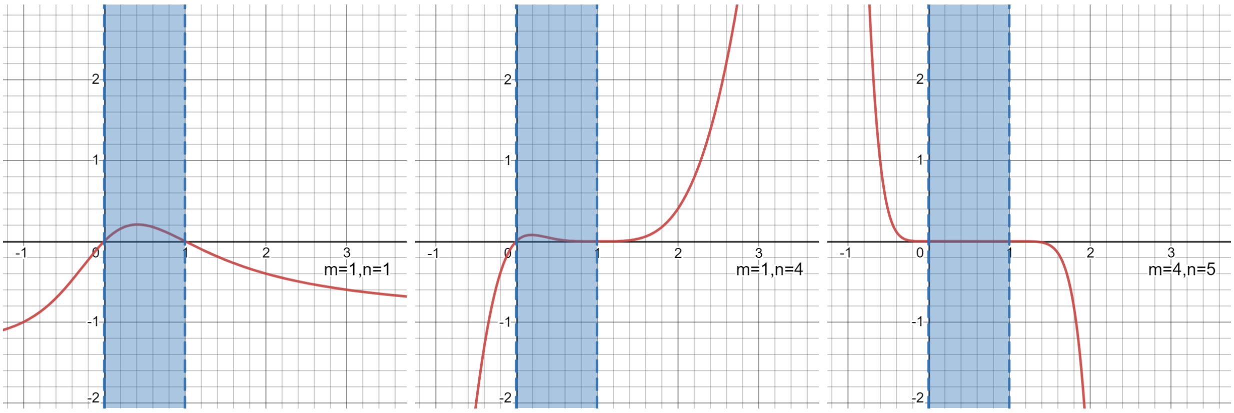

Figure 1: Graph of the integrand for various m, n

2 Want more generalised version?

N. Backhouse in [Bac95] presented a more generalised version of this integral.

$$I_{m,n} = \int_0^1 \frac{x^m(1-x)^n}{1+x^2}dx = a + b\pi + c\log(2)$$

where a, b, c are rationals that depend on $m, n \in \mathbb{Z}^+$, and a and b have opposite sign. Backhouse also showed that if $2m - n \equiv 0(\mod 4)$ then $c = 0$ and a variety of approximation to π are obtained. An integral equal to $a + b\pi$ leads to a rational approximation of π as $|\frac{a}{b}|$,

$$a + b\pi > 0 \iff \pi > -\frac{a}{b}$$

and the maximum value of the integrand gives an upper bound on the error. As m, n increase, the integrand becomes increasingly flat(*Figure-1*) (Backhouse calls them *"pancake functions"*) and the approximation to π is improved. In [Luc05], *Lucas*, showed that there are various ways to form integrals with positive integrands that evaluate to $\frac{355}{113} - \pi$.
Here I am presenting one of those integrals. Here the approach is to multiply the integrand by a low order polynomial, and adjust the coefficients to return the correct result.Choosing the simplest case where the polynomial is ≥ 0 on $[0, 1]$ *simplest in the sense of the smallest number of digits in the coefficients.* Experimenting with $I_{m,n}$ leads to the results that can be solved in the similar ways, I encourage readers to verify this.

$$\int_0^1 \frac{x^7(1-x)^7(192 - 791x + 983x^2)}{3164(1+x^2)}dx = \frac{355}{113} - \pi$$

$$\int_0^1 \frac{x^8(1-x)^8(25 + 816x^2)}{3164(1+x^2)}dx = \frac{355}{113} - \pi$$

3 Another idea to think about

$$I_{m,n} = \int_0^1 \frac{x^m(1-x)^n}{1+x^2}dx = a + b\pi + c\log(2)$$

The idea is to multiply the integrands $I_{m,n}$ by $\alpha \in \mathbb{Q}^+$ so that $b\alpha = -1$.Observe that to do this we have to set m, n in such a way that $b < 0$. hence, RHS becomes $a\alpha - \pi$, we made $c = 0$. We then require m, n large enough, so that when we add

$$\beta = \left(\frac{355}{113} - a\alpha\right) \text{ it is } > 0$$

then finally RHS becomes $\frac{355}{113} - \pi$. The integral shall look like

$$\int_0^1 \left[\frac{\alpha x^m(1-x)^n}{1+x^2} + \beta\right] dx = \frac{355}{113} - \pi$$

remember, here it's wise to chhose m, n in such way so that $4|(2m - n)$.
Try to find such an integral(not very easy!), and then calculate it.As a sample I'm giving one where

$$\alpha = \frac{1}{4}, m = 10, n = 8, \beta = \frac{5}{138450312}$$

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4 Madhava-Leibniz series:

The Leibniz's formula for π, states that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots$$

It is also called the *Madhava-Leibniz series* as it is a special case of a more general series expansion for the inverse tangent function, first discovered by the Indian mathematician *Madhava of Sangamagrama* in the 14th century, the specific case that I have written was first published by Leibniz around 1676.We shall look at one of the proofs of this series:

Proof.

$$\begin{aligned}
\frac{\pi}{4} &= \arctan(1) \\
&= \int_0^1 \frac{1}{1+x^2} dx \\
&= \int_0^1 \left(1 - x^2 + x^4 - x^6 \cdots\right) dx \\
&= \int_0^1 \left((1 - x^2 + \cdots x^{2n}) + (-1)^{n+1} x^{2n+2} (1 - x^2 + x^4 \cdots)\right) dx \\
&= \int_0^1 \left(\sum_{k=0}^{n} (-1)^k x^{2k} + (-1)^{n+1} \frac{x^{2n+2}}{1+x^2}\right) dx \\
&= \left(\sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\right) + (-1)^{k+1} \left(\int_0^1 \frac{x^{2n+2}}{1+x^2} dx\right)
\end{aligned} \tag{1}$$

Let's look at the integral at the end part. Observe that in $(0, 1)$, $0 \leq \frac{1}{1+x^2} \leq 1$ hence,

$$0 \leq \int_0^1 \frac{x^{2n+2}}{1+x^2} dx \leq \int_0^1 x^{2n+2} dx = \frac{1}{2n+3}$$

Now, we shall use *Sandwich Theorem*

$$0 \leq \lim_{n\to\infty} \int_0^1 \frac{x^{2n+2}}{1+x^2} dx \leq \lim_{n\to\infty} \frac{1}{2n+3} = 0$$

$$\implies \lim_{n\to\infty} \int_0^1 \frac{x^{2n+2}}{1+x^2} dx = 0 \tag{2}$$

Now, from (1) and using (2)

$$\begin{aligned}
\frac{\pi}{4} = \lim_{n\to\infty} \frac{\pi}{4} &= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\right) + (-1)^{k+1} \lim_{n\to\infty} \left(\int_0^1 \frac{x^{2n+2}}{1+x^2} dx\right) \\
&= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \\
&= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots
\end{aligned}$$

□

Observe that

$$\frac{\pi}{4} = 1 - \frac{1}{\binom{3}{1}} + \frac{1}{\binom{5}{1}} \cdots$$

Visualise the binomial coefficients in Pascal's triangle!

5 Nilakantha Series

$$\pi = 3 + \frac{4}{2\cdot 3\cdot 4} - \frac{4}{4\cdot 5\cdot 6} + \frac{4}{6\cdot 7\cdot 8} - \cdots$$

The formulae to calculate π had been found in India already in the fourteenth or fifteenth century. It first appeared in Sanskrit verse in the book *Tantrasangraha* from about 1500BC the Indian mathematician, astronomer and universal genius Nilakantha Somayaji $(1444 - 1544)$. unfortunately I don't know any proof of this, so I shall discuss some interesting things which relates this to inverse binomial coefficients. the modification below is *Daniel Hardisky's* modification of the series:

$$\begin{aligned}\pi &= 3 + \frac{4}{2\cdot 3\cdot 4} - \frac{4}{4\cdot 5\cdot 6} + \frac{4}{6\cdot 7\cdot 8} - \cdots \\ &= 3 + \frac{4}{6}\left(\frac{1\cdot 2\cdot 3}{2\cdot 3\cdot 4} - \frac{1\cdot 2\cdot 3}{4\cdot 5\cdot 6} + \frac{1\cdot 2\cdot 3}{6\cdot 7\cdot 8} - \cdots\right) \\ &= 3 + \frac{2}{3}\left(\frac{1}{\binom{4}{3}} - \frac{1}{\binom{6}{3}} + \frac{1}{\binom{8}{3}} - \cdots\right)\end{aligned}$$

Observe, how the inverse of binomial coefficients are behaving and giving a format of $a\pi + b$.

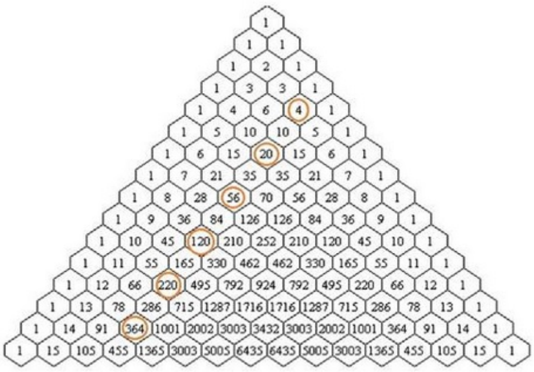

Figure 2: Pascal's triangle and Nilakantha series

6 Cherry on the top:

Some other series giving π

- This series is given by our *Srinivasa Ramanujan.*This is one of the several rapidly converging infinite series of π he found in 1910

  $$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$$

  which computes a further eight decimal places of π with each term in the series. His series are now the basis for the fastest algorithms currently used to calculate π. Even using just the first term gives

  $$\pi \approx \frac{9801}{2206\sqrt{2}} \approx 3.14159273$$

  There is a generalization of this called *Ramanujan-Sato series.*

- want to count π upto crazy high decimal places? Extremely long decimal expansions of π are typically computed with iterative formulae like the *Gauss–Legendre algorithm* and *Borwein's algorithm*.
- Want to count ta particular number digit of π here .

References

[Bac95] Nigel Backhouse. "Pancake Functions and Approximations to π". In: *The Mathematical Gazette* 79.485 (1995), pp. 371–374. DOI: https://doi.org/10.2307/3618318.

[Luc05] Stephen Lucas. "Integral proofs that $355/113 > \pi$". In: *Gazette of the Australian Mathematical Society* 32.263 (2005), pp. 263–266.