What is modern geometry about?

Vamsi Pritham Pingali

Indian Institute of Science

Fields of modern geometry

- Differential geometry and generalisations (Distances on general objects).
- Algebraic geometry (Zeroes of polynomials).
- Topology (Study of shape without taking distances, angles, lines, etc into account).
- Convex geometry and Linear algebra (Study of objects where the notion of a straight line makes sense).
- Discrete and Computational geometry (Step-by-step procedures to do the above).
- We will focus on the first.

Non-modern geometry

- Geometry has a rich history dating back to around 2000 B.C (Egyptian, Indian, and Babylonian civilisations. (Later, Greek, Arab, and Chinese.))
- Early geometry focussed on measuring things, i.e., lengths, areas, volumes, and relations between them.
- It was "applied" to astronomy, navigation, and related subjects.
- While the statement of the Pythagorean theorem was known earlier, the first proof (by areas) was due to the Pythagorean school.
- Primitive trigonometry existed quite early too.
- Euclid revolutionsed the subject by introducing axioms.
- The Italian renaissance artists motivated projective geometry.

Euclidean geometry

- Euclid treated points, lines, and planes as primitive/undefined notions and postulated 5 properties about them (Just as ZFC is about sets.)
- By modern standards, his axioms are not rigorous. Hilbert postulated 20 axioms for Euclidean geometry. (Three primitive relations are betweeness, incidence, and congruence.)
- Using just these axioms, one can define real numbers! (See Foundations of geometry by Borsuk and Szmielew.)
- Euclid's fifth postulate is "Through every point not lying on a line, there is a unique parallel line". This postulate was long thought to be a consequence of the other postulates.

Non-Euclidean geometry

- Euclid's fifth postulate was found by Gauss and Lobachevsky to be independent of the other postulates in the sense that they found geometries where it was violated. Non-Euclidean geometry is the study of distances and angles where one or more of Euclid's axioms are violated.
- The sphere is the simplest example. (The sum of angles of a triangle is certainly larger than 180 degrees.)
- The analogues of standard theorems like sine rule, etc have already been proven on many of these spaces. For instance, in hyperbolic geometry, similarity of triangles is a trivial concept. Indeed, it turns out that the sum of angles of a triangle is <180 degrees. Therefore, drawing the two similar triangles, we can show that they are congruent.

Non-Euclidean geometry

- What is more interesting is the question of congruence (when are two spaces with ostensibly different ways of measuring distances and angles secretly the same)? and "what is the best way of measuring distances and angles on a given space"?
- The latter question might seem stupid but as Gauss proved, there is no way to draw a map of any part of the earth (to scale) on a piece of paper. (Meaning that the "best" way (whatever that means) of measuring distances on the sphere cannot be the usual Euclidean one.)
- A large portion of research on geometry (Riemannian geometry) focusses on the latter question (typically, "best" means "nicest curvature properties"). The Poincaré conjecture can be viewed as a special case of such a question. So can the (less) famous Calabi conjecture that fetched Yau the fields medal.

A crash course in differential geometry of curves and surfaces

- Differential geometry studies geometry on curved objects. The very definition of curvature is not trivial.
- Suppose $\gamma(t)$ is a smooth unit-speed curve in $\mathbb{R}^3$. Then $\|\gamma''\|$ is called the *curvature* of the curve. For circular motion, the centripetal acceleration is $\frac{1}{R}$. So the "curvature" of a large circle (close to a straight line) is small and that of a small circle is large.
- Suppose $S \subset \mathbb{R}^3$ is a smooth surface, i.e., near every point on S, we can parametrise S as $\vec{r}(u,v)$ such that $\vec{r}_u \times \vec{r}_v \neq \vec{0}$, i.e., the unit normal makes sense.
- Note that if $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function such that $\nabla f \neq 0$ for all points on $S = f^{-1}(0)$, then S is a smooth surface (using the implicit function theorem).

Gaussian curvature

- Our aim is to define the curvature of S at every point $p \in S$.
- Take all unit-speed curves γ on S passing through p. Consider $\gamma''.\vec{N}$ at p. The product of the maximum and minimum such numbers is called the Gaussian curvature of S at p.
- Since $\gamma''.\vec{N} = -\gamma'.\vec{N}' = -\gamma'.\nabla_{\gamma'}\vec{N}$, and $\nabla_{\vec{a}}\vec{N}$ is tangent to S whenever $\vec{a}$ is, the map $\vec{a} \to \nabla_{\vec{a}}\vec{N}$ is a linear map taking tangent vectors to tangent vectors. The Gaussian curvature is simply the determinant of this "shape operator". (The trace of this map is called the mean curvature and it is zero when the surface locally minimises its area.)

Examples

- The Gaussian curvature of the plane $z = 0$ is 0.
- Take a sphere $x^2 + y^2 + z^2 = R^2$. $\vec{N} = (x/R, y/R, z/R)$. At a point (x_0, y_0, z_0), choose coordinates x, y and tangent vectors $(1, 0, -\frac{x_0}{z_0})$ and $(0, 1, -\frac{y_0}{z_0})$. Now the matrix for the shape operator is $\frac{1}{R} Id$. Thus its determinant is $\frac{1}{R^2}$.
- The cone $z^2 = x^2 + y^2$: Now, $\nabla f = (2x, 2y, -2z)$ which is non-zero away from the vertex. $\vec{N} = (x/\sqrt{2}z, y/\sqrt{2}z, 1/\sqrt{2})$. Take tangent vectors $(1, 0, \frac{x_0}{z_0})$ and $(0, 1, \frac{y_0}{z_0})$. The determinant of the shape operator turns out to be 0. The key point is the sharp corner at the vertex.

The theorema egregium

- The remarkable thing ("egregium") is that this number depends *only* on the infinitesimal distances on the surface and not on how the surface "sits" in $\mathbb{R}^3$, i.e., suppose we take a part of a sphere made of an unbreakable thin material and we deform it smoothly any way we want, the curvature remains the same.
- More precisely, suppose $(x(u,v), y(u,v), z(u,v))$ and $(x'(u,v), y'(u,v), z'(u,v))$ are two such surfaces such that the "infinitesimal length element" $dx^2 + dy^2 + dz^2 = dx'^2 + dy'^2 + dz'^2$, then their Gaussian curvatures are equal.
- Since the sphere has positive curvature and the plane has 0 curvature, we cannot possibly draw a faithful map of any part of the sphere on a plane (that is, any way to wrap a paper around a sphere causes distortions).

The area element and the Gauss-Bonnet theorem

- In general, given a surface S, one can measure distances by putting *any* metric $h = Edu^2 + 2Fdudv + Gdv^2$ on it. If we consider the infinitesimal parallelogram with vertices $(u, v), (u + du, v), (u + du, v + dv), (u, v + dv)$, after a rotation of coordinates, WLOG we may assume that at (u, v), $F = 0$. Thus, the area is $dA = \sqrt{E}\sqrt{G}dudv = \sqrt{\det(h)}dudv$.
- Every compact (closed and bounded) surface without boundary turns out to be a doughnut-like object with g holes (where g is called the genus). Now one can write coordinates that parametrise almost all of such a surface except for a few curves. One can then define the integral $\int\int KdA$ over the domain of (u, v).
- It turns out that this integral is $2\pi(2 - 2g)$ regardless of the metric h! This fact is called the Gauss-Bonnet theorem.

The uniformisation theorem

- So on a torus, where $g = 1$, the integral is 0. Therefore, there is no metric on the torus whose curvature is a constant $= 1$ everywhere for instance.
- The uniformisation theorem: On a surface of genus $g \geq 2$, there exists a metric whose curvature is -1, when $g = 1$, there is one with $K = 0$, and when $g = 0$, there is one with $K = 1$.
- One way to prove this theorem is by recasting it as a PDE.
- While $g = 0$ appears easiest (because we already know the metric on a sphere!), in higher dimensions, this is the hardest case! (There is more than one higher-dimensional generalisation: The Yamabe problem and the Yau-Tian-Donaldson conjecture.)

Speaking of PDE: Conservative vector fields

- The following slides are simply a bonus and have little to do with the rest!
- Recall that if a vector field $\vec{F}$ is conservative, (which is the same as $\vec{F} = \nabla\phi$) then $\nabla \times \vec{F} = \vec{0}$. What about the converse?
- Not true! Let $\vec{F} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$. $\int_{Circle} \vec{F}.\vec{dr} \neq 0$!
- This $\vec{F}$ is actually the magnetic field circling a current-carrying wire coming out of the plane at the origin.
- The problem here is that the *domain* of $\vec{F}$ has a "hole" in it (the origin). In other words, studying PDE ("Is $\vec{F} = \nabla\phi$ if $\nabla \times \vec{F} = \vec{0}$") can tell us about the shape of a domain.
- In fact, one can prove that on $\mathbb{R}^2-$ a bunch of points, whenever $\nabla \times \vec{F} = \vec{0}$, if we *subtract* a bunch of magnetic fields, then the rest is a gradient! (A little difficult! - See "The Beautiful Mind").

A fancier version

- A vast generalisation of this concept (both in the kinds of domains (can be things like a sphere and a doughnut) and a generalisation of a vector field and a cross product) is called de Rham cohomology (a part of differential topology).
- John Milnor won a fields medal for showing that there is more than one way to do calculus on a 7-dimensional sphere. Simon Donaldson won it for showing a theorem which when combined with another fields medallist's work (Michael Freedman) shows that there is more than one way to do calculus on $\mathbb{R}^4$.

A fancier version

- Ed Witten (the only physicist to win the fields medal) and Nathan Seiberg later gave a simpler proof of Donaldson's results.
- The point is that Donaldson looked at all possible solutions of a certain PDE (called Yang-Mills equations) and studied the structure of such a space. (A part of that work involved the Atiyah-Singer index theorem (fields medal to Atiyah and Abel to Atiyah-Singer) and the work of Uhlenbeck (Abel prize).)

Thank you