FRACTAL GEOMETRY
OF NATURE

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Geometry deals with two kinds of entities:

- objects: points, circles, lines, curves, cylinders, tetrahedrons,
- spaces: in which the objects live.

In geometry we generally study

- idealized objects: triangles, circles, spheres and rectangular parallelopipeds.

- idealized spaces: Euclidean.

The narrow confines of Euclidean spaces was broken in the 19th Century. But objects still remained Euclidean.

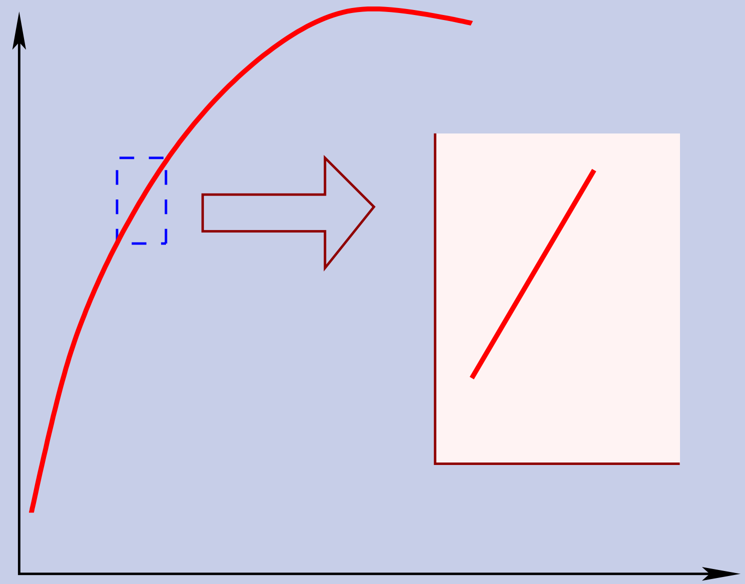
But *Nature* abounds in irregular objects.

Mountains are not cones,
clouds are not spheres,
lightnings are not straight lines.

Towards the end of the twentieth century we broke out of the compartment of Euclidean objects. Geometers are now considering these irregular objects as valid subjects of study. And that is what fractal geometry is all about.

What distinguishes natural objects from idealized objects?

In a geometrical object
like a curve, the closer you look
the more it loses structure.



When that happens, the derivative can be defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

This limit would exist only if the curve smoothens out into the tangent as $\Delta x \to 0$

Natural objects never flatten out — at whatever level of magnification you may look at them.

Think of

- your skin,
- a tree trunk,
- curve for share prices or exchange rates,
- load on a power plant,
- a coastline.

These geometrical objects are continuous but not differentiable anywhere.

They reveal structures within structures as you zoom closer.

How long is the coastline of England?

Measured length of the coastline depends on the yardstick of measurement.

In the limit, when the yardstick length shrinks close to zero, the length of the coastline becomes infinite.

Yet the area of England *is* finite. Thus the coastline is a curve of infinite length enclosing a finite area.

Same is the case of all 3-D natural objects enclosed by natural surfaces.

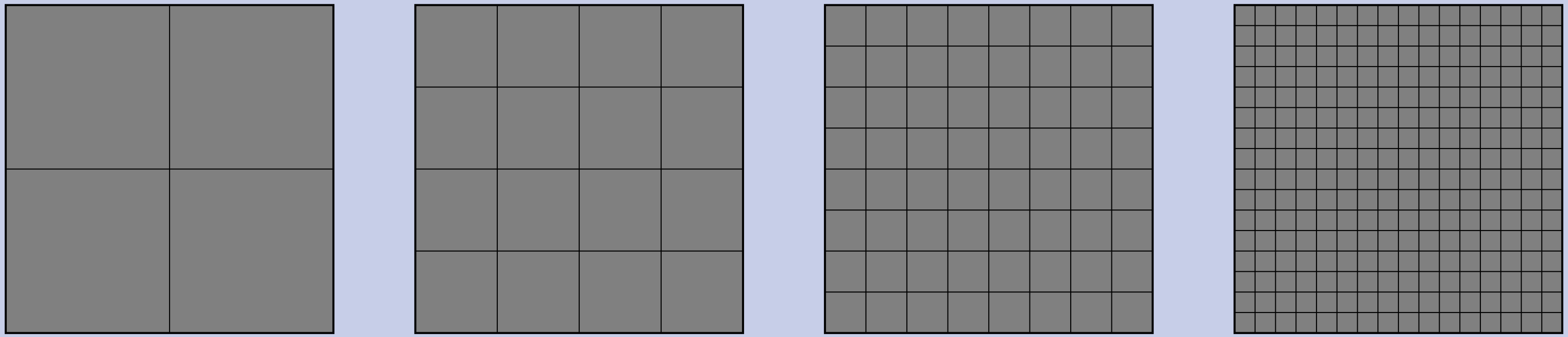
Example: our lungs.

These objects, evidently, need a new mathematical tool for characterization.

Dimension

- The dimension of an object and that of the embedding space are different.
- The dimension of the embedding space is given by the degrees of freedom.
- The dimension of an object has to be defined according to the way *it fills space.*

Take an Euclidean object: the square. How does it fill space?

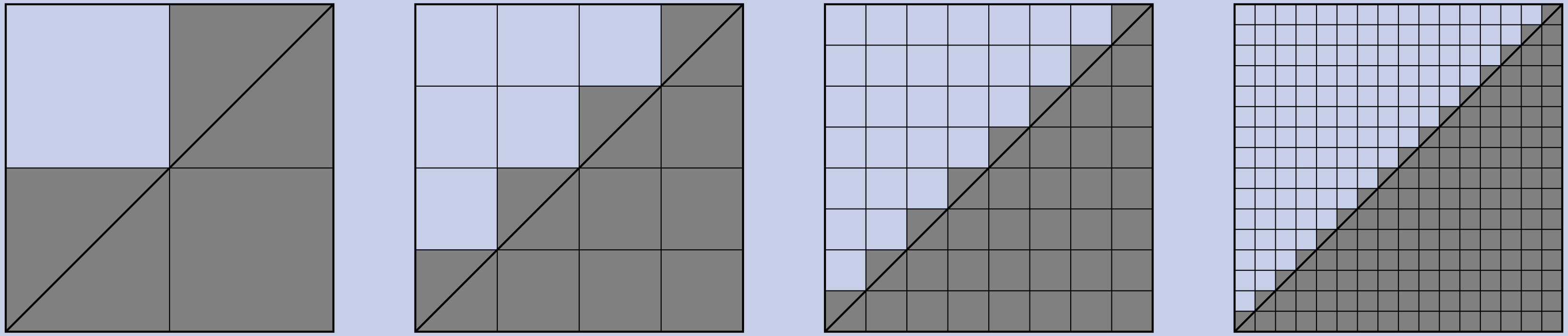


$$N(\epsilon) = \left(\frac{1}{\epsilon}\right)^2$$

where

ϵ is the grid length, and

$N(\epsilon)$ is the number of grid elements required to cover the object.



For a right angled triangle, $N(\epsilon) \to \frac{1}{2}\left(\frac{1}{\epsilon}\right)^2$

For a circle, $N(\epsilon) \to \frac{\pi}{4}\left(\frac{1}{\epsilon}\right)^2$

To generalize, we can write $N(\epsilon) \to K\left(\frac{1}{\epsilon}\right)^2$

We can extract the dimension (2 in this case) from it as follows:

$$\ln N(\epsilon) = \ln K + 2 \ln \frac{1}{\epsilon}$$

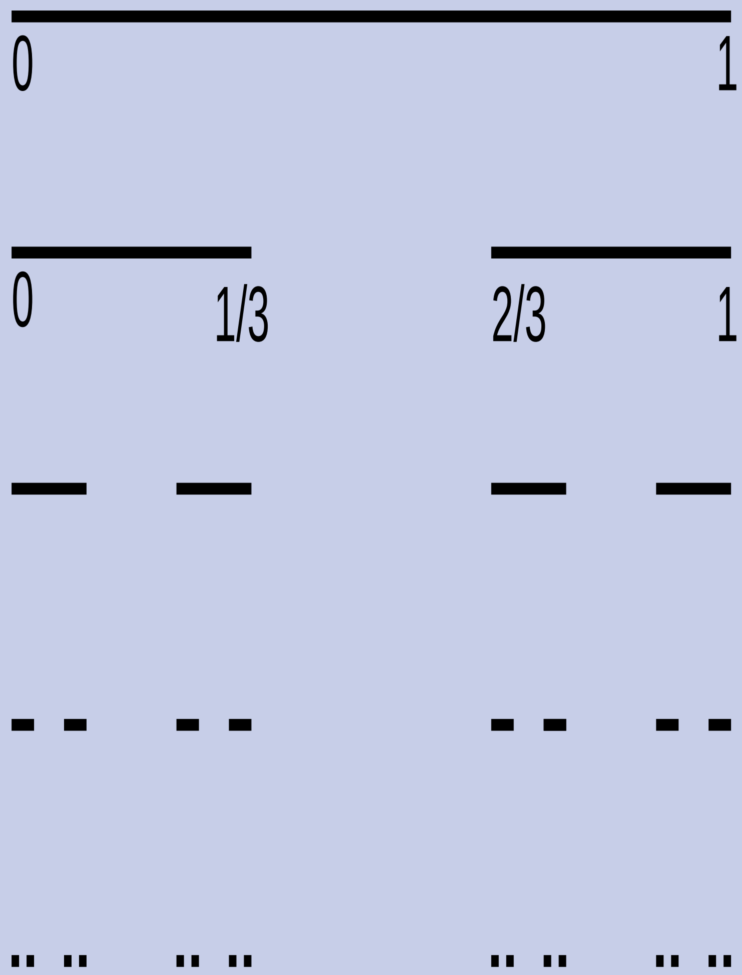
$$2 = \frac{\ln N(\epsilon)}{\ln \frac{1}{\epsilon}} - \frac{\ln K}{\ln \frac{1}{\epsilon}}$$

The second term would vanish as $\epsilon \to 0$. Thus the dimension D of the object is given by

$$D = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln \frac{1}{\epsilon}}$$

The Cantor set:

- Begin with a line [0,1].
- First iterate: Remove the middle one-third.
- Second iterate: Remove the middle one-third of the remaining segments.
- Continue infinitely.

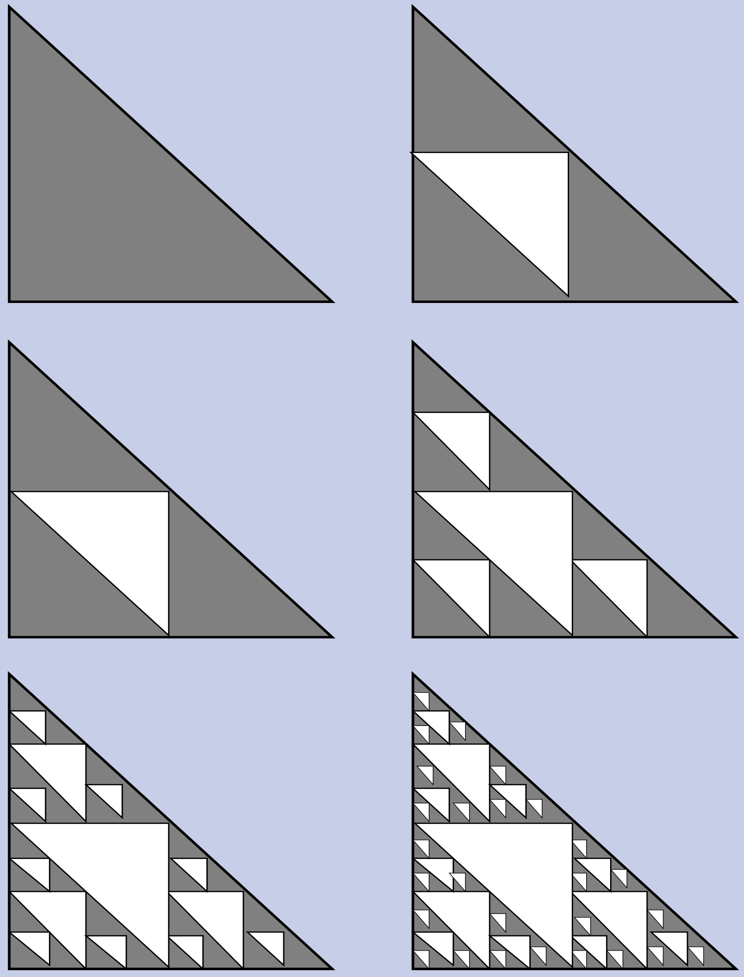


Dimension:

$$D = \frac{\ln 2}{\ln 3} = 0.63\cdots$$

The Sierpinski triangle:

- Begin with a right-angled triangle.
- First iterate: Remove the middle one-third.
- Second iterate: Remove the middle one-third of the remaining segments.
- Continue infinitely.

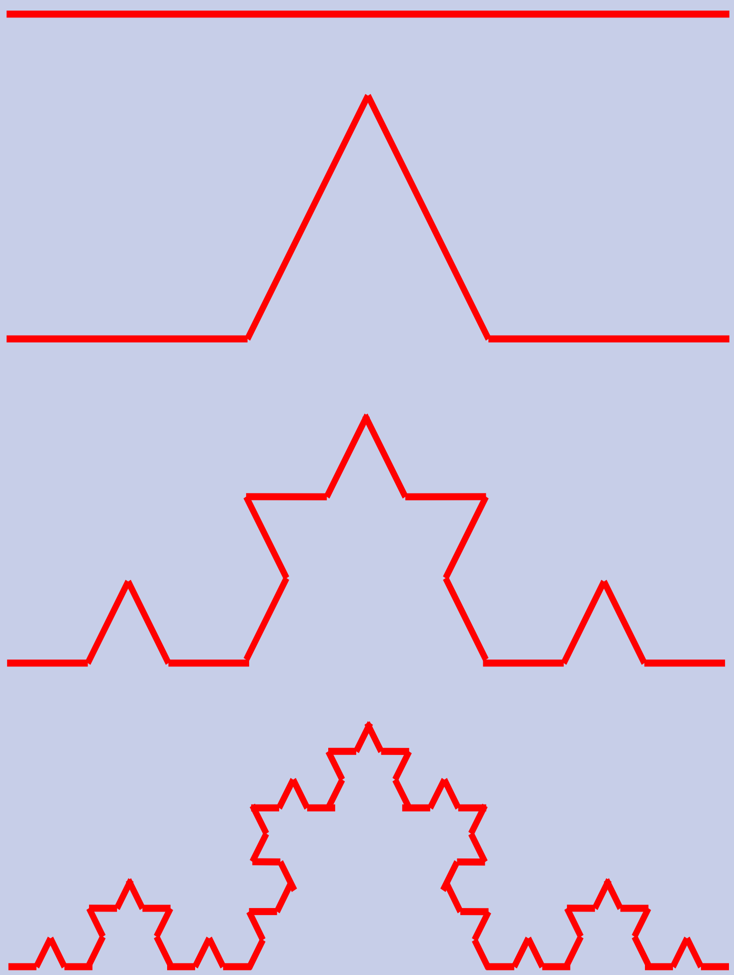


Dimension:

$$D = \frac{\ln 3}{\ln 2} = 1.5849625\cdots$$

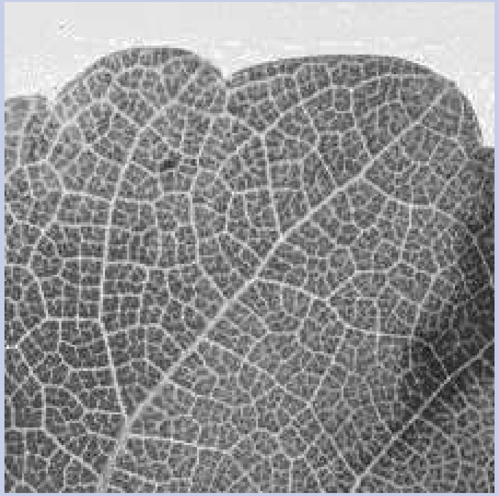
The Koch curve:

- Begin with a line [0,1].
- First iterate: Remove the middle one-third, and add two equal segments to make a triangle.
- Second iterate: Remove the middle one-third of the remaining segments, and add new segments.
- Continue infinitely.

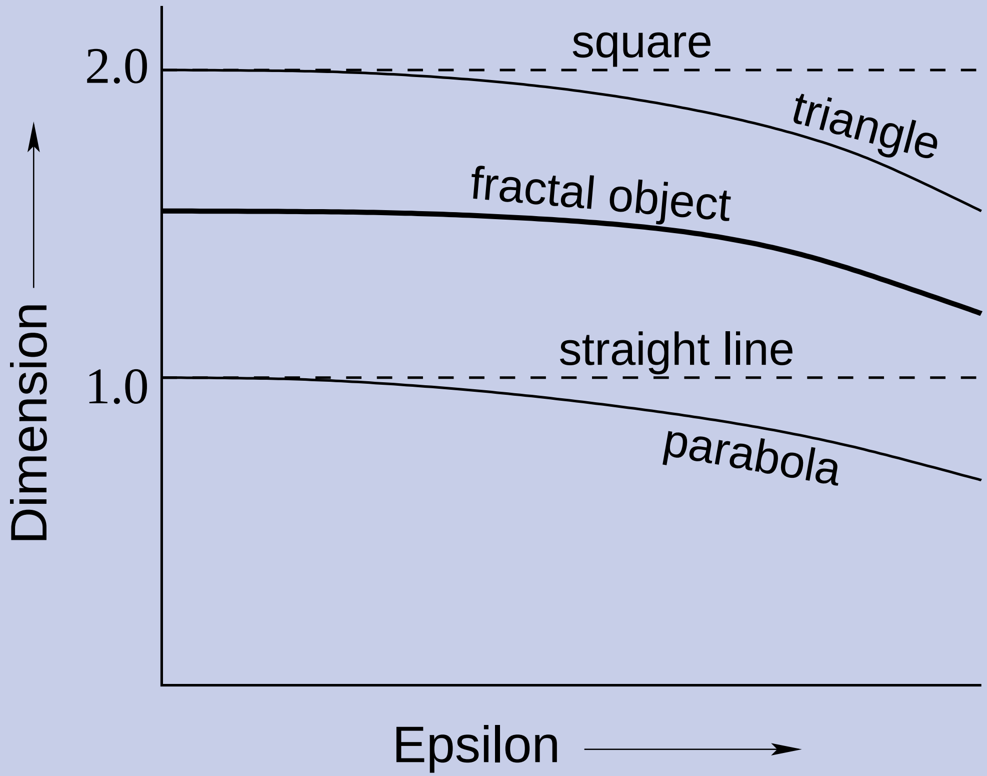


Dimension:

$$D = \frac{\ln 4}{\ln 3} = 1.26\cdots$$

If we subject a natural object to the above procedure, the dimension turns out to be a fraction.



Geometrical objects with fractional dimensions are called *Fractals*.

What use is fractal dimension?

It actually quantifies the surface characteristics.

- Carbon particles in automobile exhaust,
- Electrical contact between surfaces,
- Silt particles in river water,
- Grain boundaries in metals, semiconductors or alloys,
- Rock structures.
- Solid catalysts and electrodes,
- The veins in the plant leaves, the arteries and veins in your body, tumours and cancer cells, etc.

How is fractal geometry relevant in dynamics?

Generally the models representing realistic dynamical systems may be quite complicated. But simplified versions often reveal properties observed in complicated systems as well.

Example:

$$Z_{t+1} = (Z_t)^2 + C$$

where Z_t is the state of the system at the t-th instant,
Z_{t+1} is the state at the next instant.

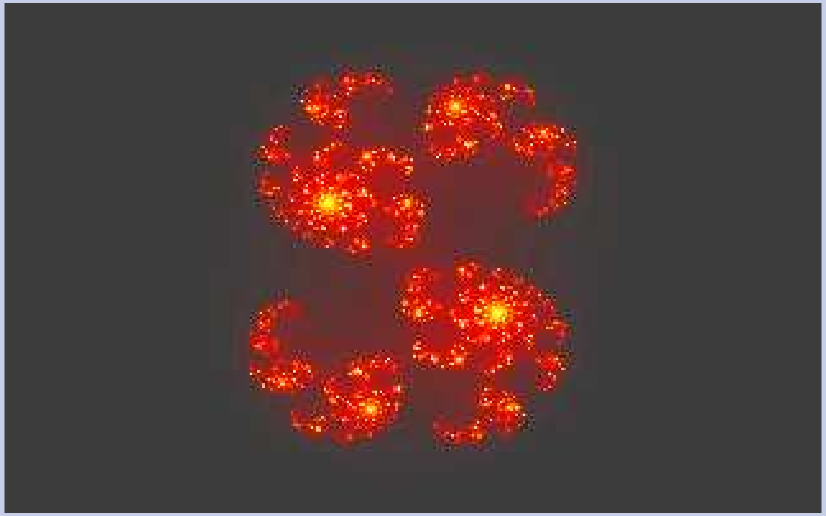
The Z's and C are complex numbers.

$\mapsto$ a 2-D system represented by a single equation.

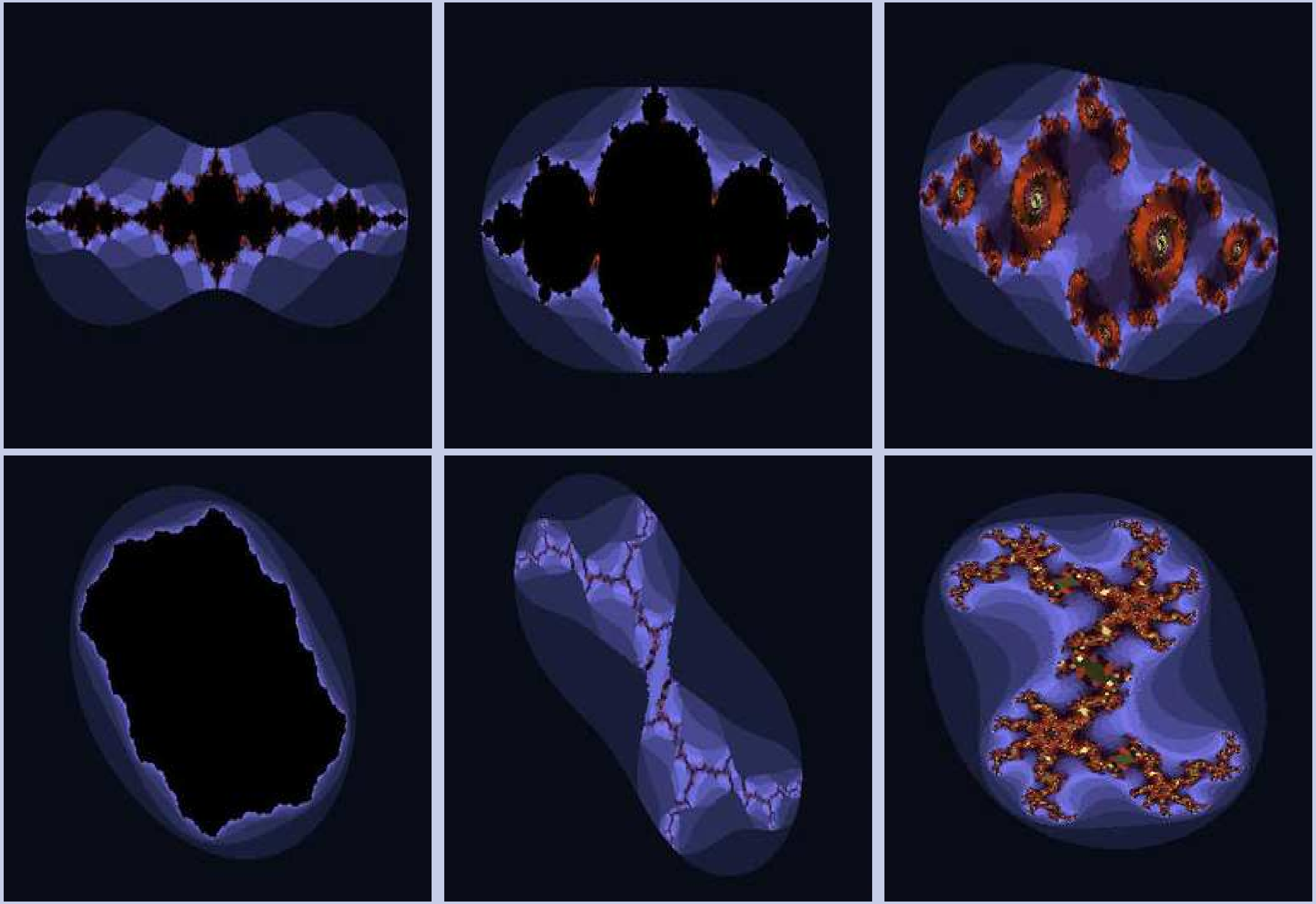
- ➜ Choose a value of C and a starting point Z_0,
- ➜ Calculate subsequent values of Z and observe the dynamics.

For some values of Z_0 the system remains bounded, for some other values Z increases without bounds, i.e., runs towards infinity.

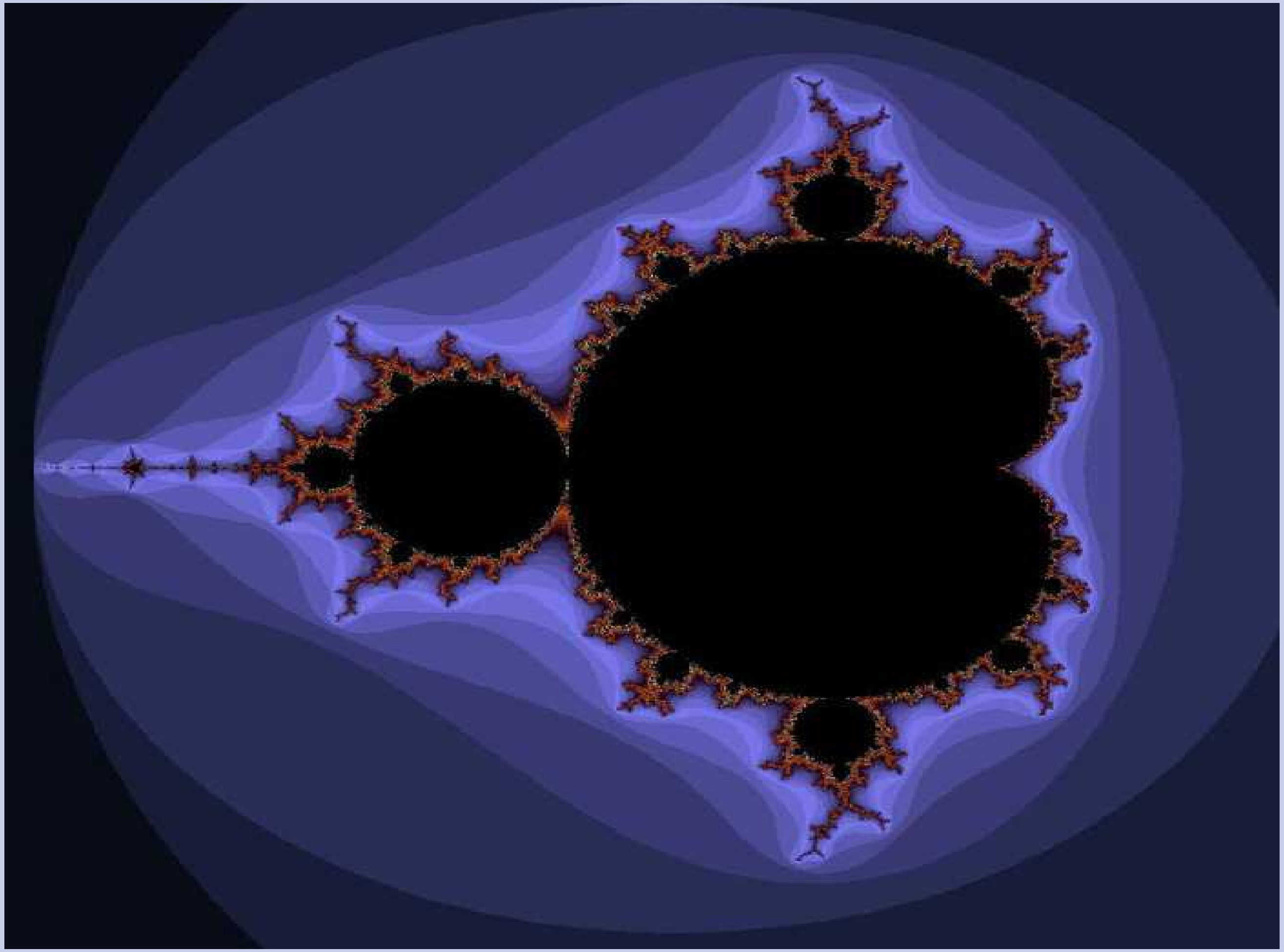
Plot those values of Z_0 for which the system remains within bounds — the Julia Set.



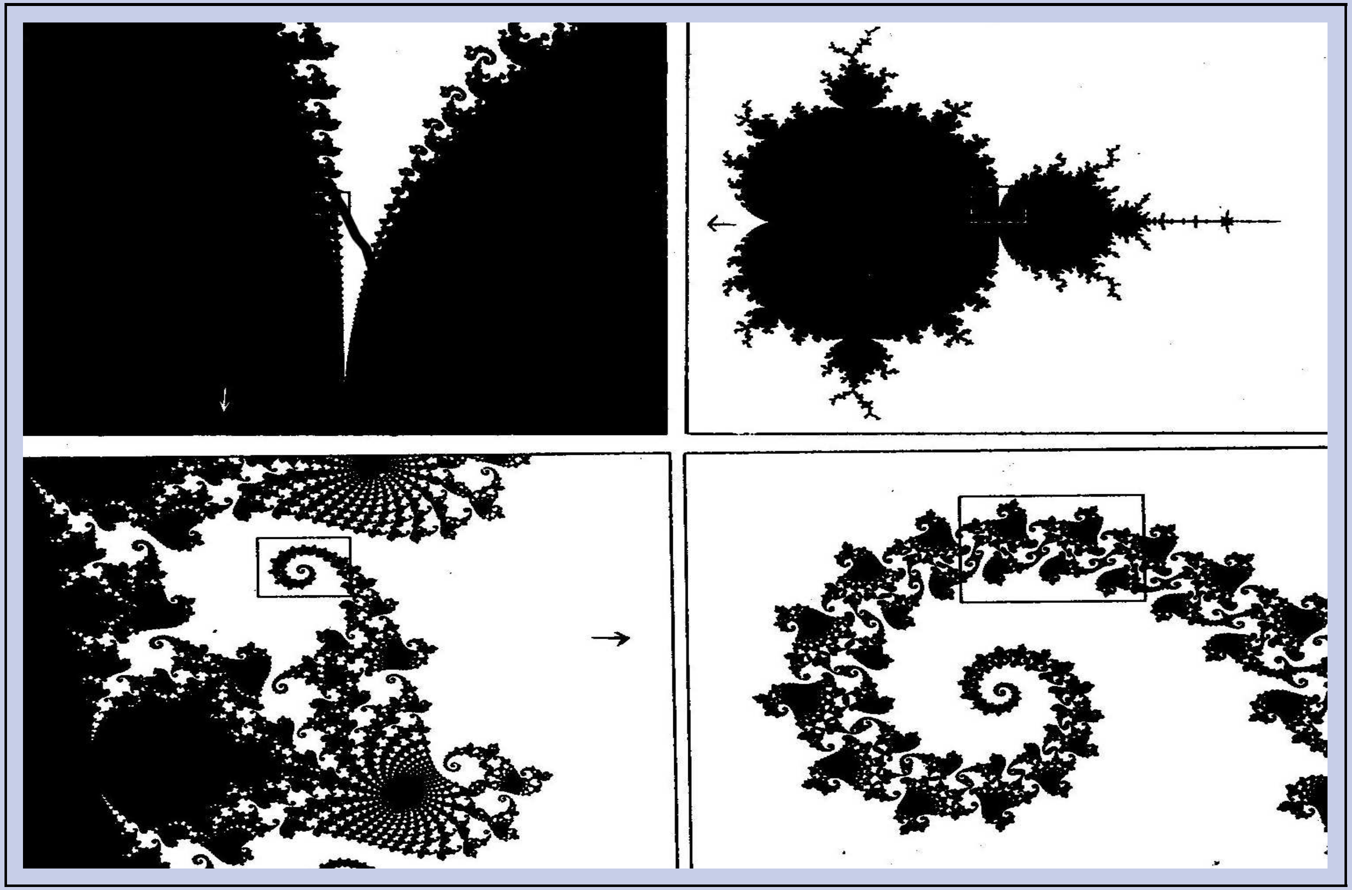
For different values of C you get different sets:



Now vary the parameter C and keep the initial condition Z_0 fixed $\rightarrow$ Mandelbrot set.

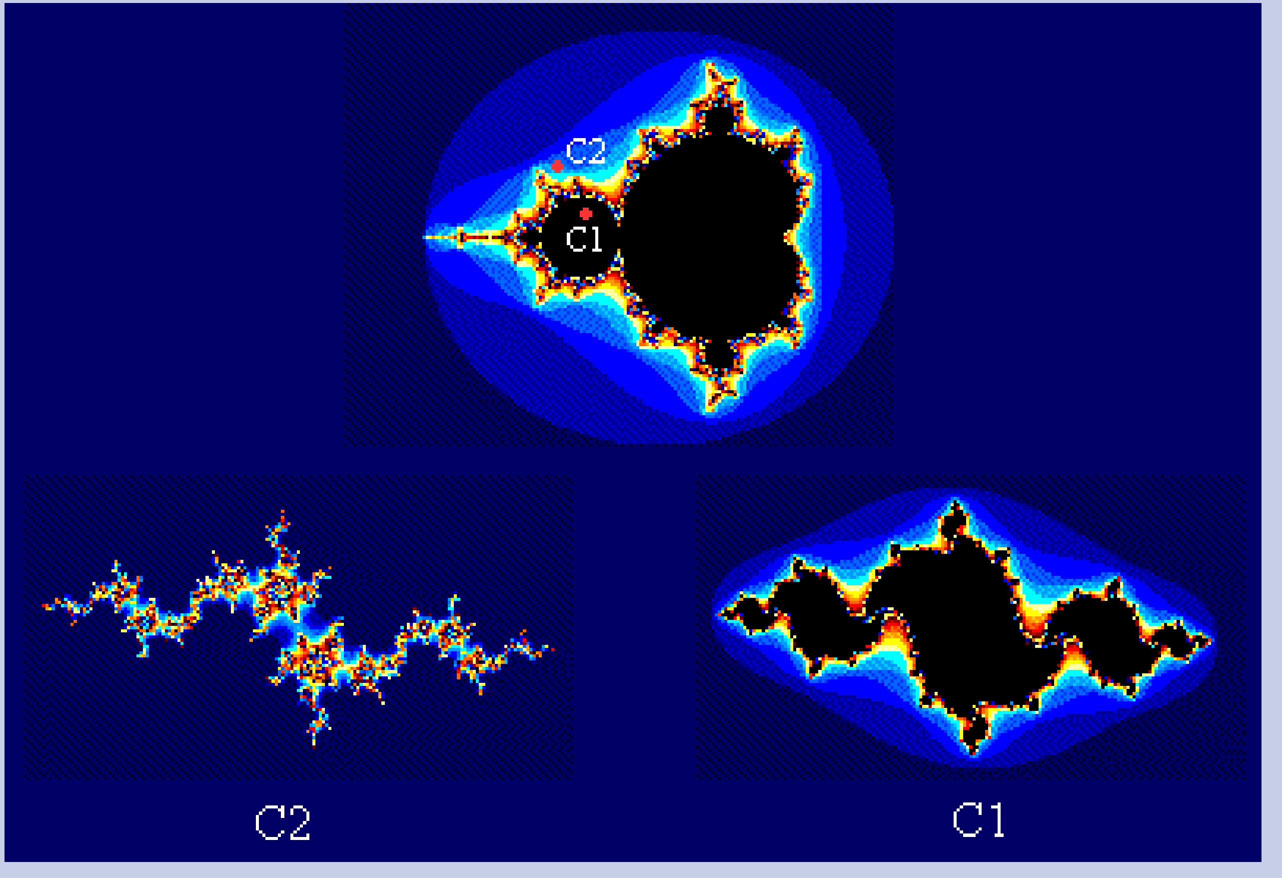


However small part of the figure you may want to observe, it reveals rich internal structures. The same is true for Julia sets.



Julia set	$\rightarrow$	fractal in phase space,
Mandelbrot set	$\rightarrow$	fractal in parameter space.

There is a Julia set for every point in a Mandelbrot set.



Do such things happen in practical systems?

Example:

Swing of a ship in ocean.

Small angle of tilt $\rightarrow$ return to upright position,

Large angle of tilt $\rightarrow$ capsize.

The equation of motion is

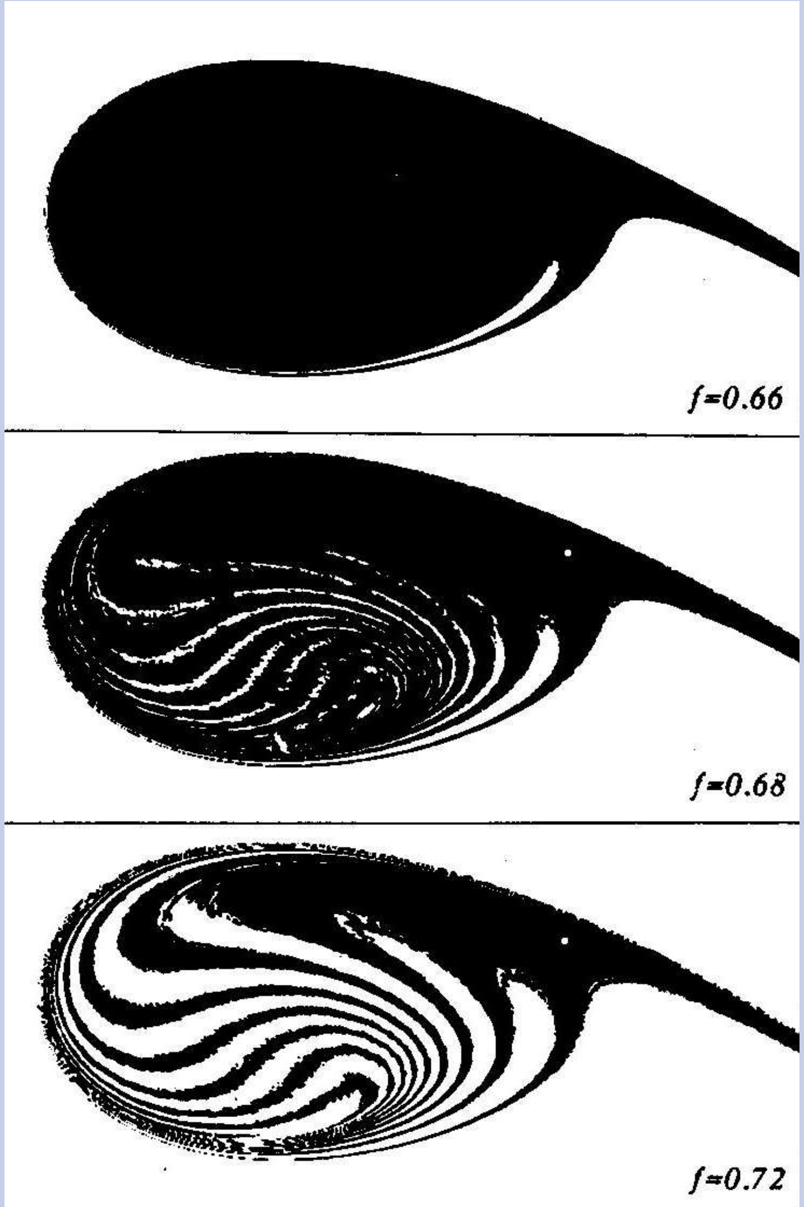
$$\ddot{x} + \beta\dot{x} + x - x^2 = F \sin \omega t$$

where β represents the frictional damping and waves of intensity F strike the ship with a frequency ω.

Black:
initial conditions for stability
White:
initial conditions for capsize

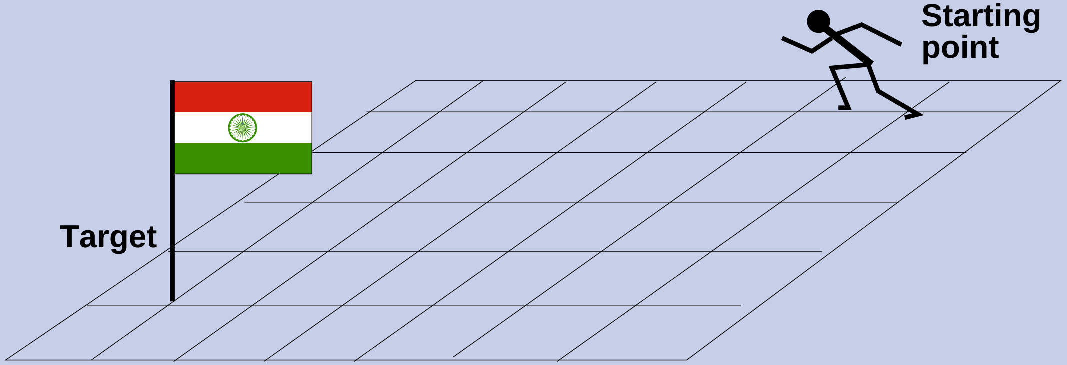
Fixed parameters:
$\beta = 0.1$, $\omega = 0.85$,
Variable parameter:
(a) $F = 0.66$,
(b) $F = 0.68$,
(c) $F = 0.72$.

The basins of attraction are fractal.



The space where images live

- Any black-and-white image is nothing but a set of points—a compact subset of $I\!R^2$.
- Define a new *space* whose elements are such compact subsets of the $I\!R^2$ space.



You can reach the target provided

- The space is defined
- We can take steps in this space (an operation in the space that takes one point to another)
- Repeated stepping defines a sequence
- The sequence is convergent
- The space is complete

- Can we define a metric (distance) between two points of this space (difference between two images)?

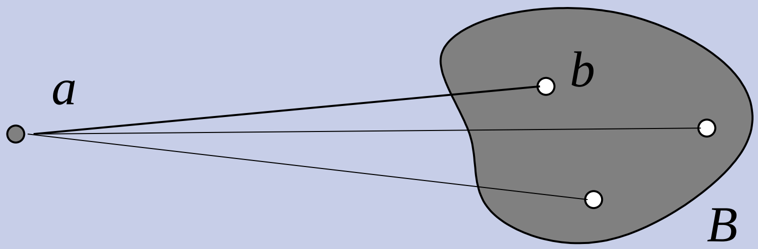
  Conditions of a metric:

  1. $d(a, b) \geq 0$
  2. $d(a, b) = 0$ iff $a = b$
  3. $d(a, b) = d(b, a)$
  4. $d(a, b) + d(b, c) \geq d(a, c)$ (triangle inequality)

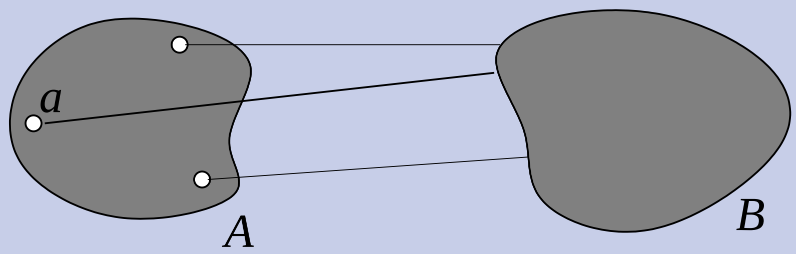
Distance between a point and another point: Euclidean distance:

$$d(a,b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

Distance between a point and a set: $d(a, B) = \min\{d(a,b),\ b \in B\}$



Distance between a set and a set: $d(A, B) = \max\{d(a,B),\ a \in A\}$

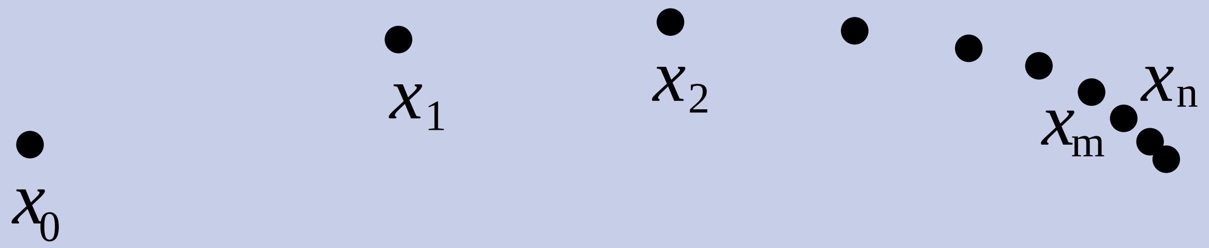


But this may not satisfy $d(a,b) = d(b,a)$.

Hence $h(A,B) = d(A,B) \vee d(B,A) \quad \longrightarrow$ Hausdorff metric.

The space where images live is a metric space.

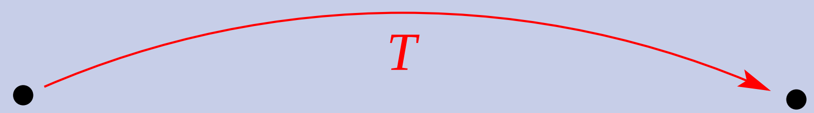
Cauchy sequence: Let (X, d) be a metric space. The sequence $\{x_n\} \in X$ is said to be a Cauchy sequence if, for every arbitrarily small number ϵ, there exists a positive number N such that $d(x_m, x_n) < \epsilon \ \forall \ m, n > N$.



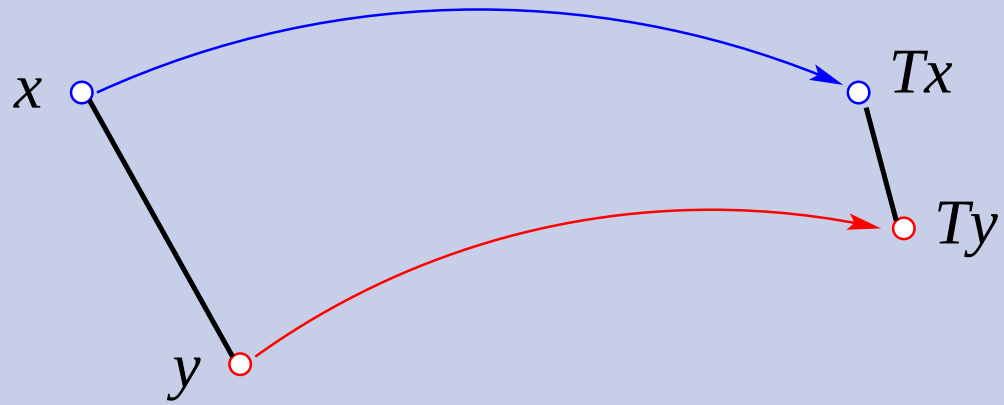
Known: Cauchy sequences converge to a fixed point.

Can we define a Cauchy sequence in the space of images?

A mapping: $T : X \to X$ takes an element of the space X to another element.



Contraction mapping



A mapping is defined as a contraction mapping if $d(Tx, Ty) \leq sd(x, y), 0 < s < 1, \ \forall x, y \in X$.

Banach's contraction mapping theorem: Let (X, d) be a complete metric space, and let $T : X \to X$ be a contraction mapping. Then repeated application of T defines a sequence that converges on a unique fixed point in X.

Proof: Consider an arbitrary $x_0 \in X$.

Consider the sequence $x_0, x_1, x_2, x_3, \cdots, x_m, x_{m+1}, \cdots, x_n \cdots$

$$\begin{aligned} d(x_m, x_{m+1}) &= d(Tx_{m-1}, Tx_m) \\ &\leq sd(x_{m-1}, x_m) \quad \text{(since } T \text{ is contractive)} \\ &\leq s^2 d(x_{m-2}, x_{m-1}) \\ &\vdots \\ &\leq s^m d(x_0, x_1) \end{aligned}$$

Now consider $d(x_m, x_n)$. By triangle inequality,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) \cdots d(x_{n-1}, x_n) \\ &\leq s^m d(x_0, x_1) + s^{m+1} d(x_0, x_1) + \cdots s^{n-1} d(x_0, x_1) \\ &\leq d(x_0, x_1)[s^m + s^{m+1} + \cdots s^{n-1}] \\ &\leq d(x_0, x_1) s^m [1 + s + s^2 \cdots s^{n-m-1}] \\ &\leq d(x_0, x_1) s^m [1 + s + s^2 \cdots \infty] \\ &\leq d(x_0, x_1) \frac{s^m}{1-s} \end{aligned}$$

Since $0 < s < 1$, $d(x_m, x_n) \to 0$ as $m, n \to \infty$
$\Rightarrow \{x_n\}$ is a Cauchy sequence
$\Rightarrow \exists\, x^* \in X$ such that $x_n \to x^*$, the fixed point of T.

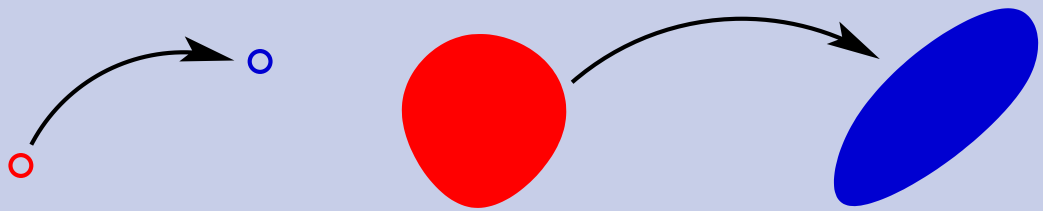
So, we need to define a contraction mapping in the space of images.

- We define a mapping that takes a point to another point. The simplest is the affine mapping:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

- The affine mapping is contractive if the matrix has eigenvalues less than unity.

- Applied to all the points in a set, it maps an image to another, i.e., from an element of the Hausdorff space to another.



Is the resulting mapping contractive? Yes.

Proof: Let $B, C \in \mathcal{H}(\mathbf{X})$. Then

$$\begin{aligned} d(w(B), w(C)) &= \max\{\min\{d(w(x), w(y)) : y \in C\} : x \in B\} \\ &\leq \max\{\min\{s \cdot d(x, y) : y \in C\} : x \in B\} \\ &\leq s \cdot d(B, C) \end{aligned}$$

Similarly, $d(w(C), w(B)) \leq s \cdot d(C, B)$.

$$\begin{aligned} h(w(B), w(C)) &= d(w(B), w(C)) \vee d(w(C), w(B)) \\ &\leq s \cdot h(B, C) \end{aligned}$$

One can also define a more general mapping in the Hausdorff space as a collection of affine transformations:

$$W = \{w_1, w_2, w_3 \cdots w_n\}$$

If X is a set in $I\!R^2$ (an image), then

$$W(X) = U_{j=1}^{n} w_j(X)$$

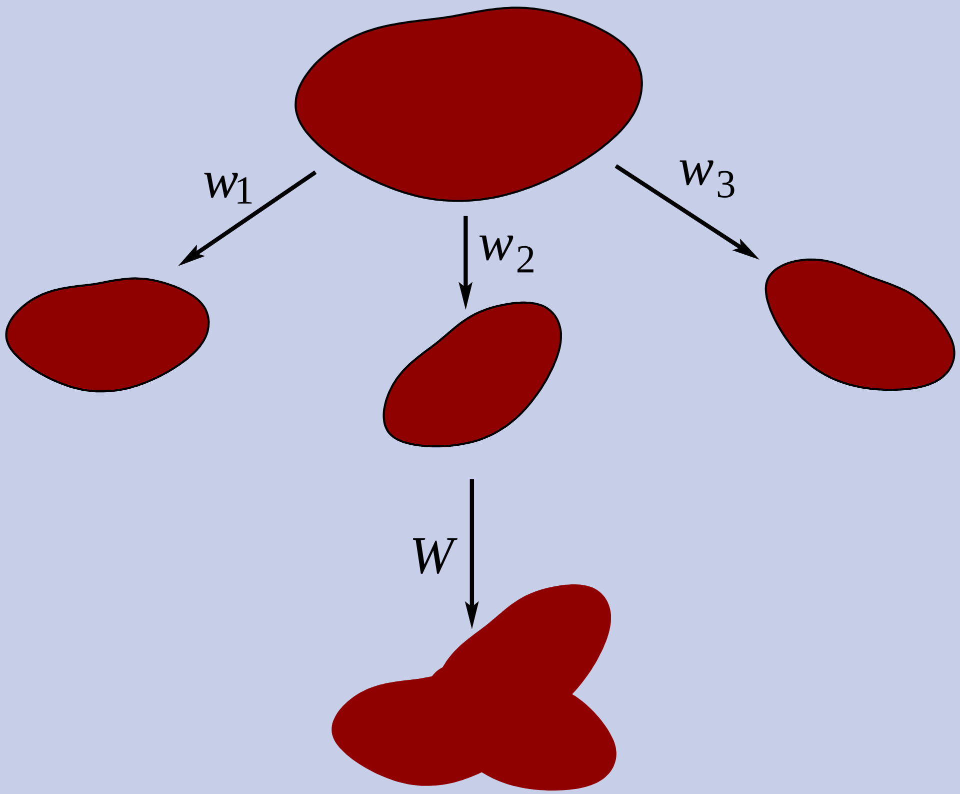
We can iterate the transformation W on a figure to get a sequence of figures.

We thus obtain a sequence of elements in the Hausdorff space.

The set of functions W is called 'Iterated Funtion System' (IFS).

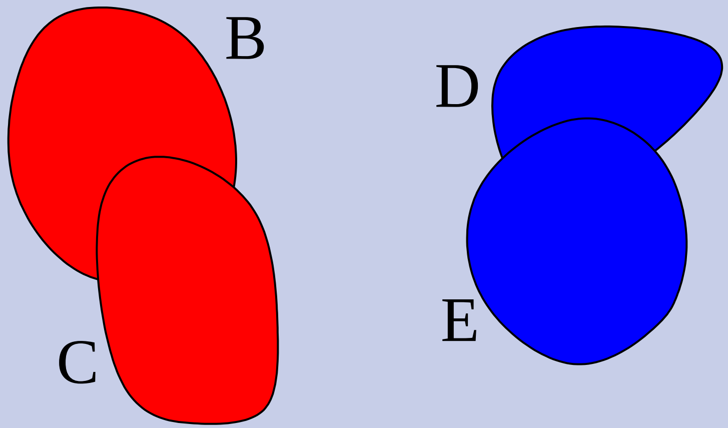
- Will such a sequence lead us anywhere?
- No, unless the sequence is convergent.

- If n number of such affine transforms are defined, the resulting image is the union of the ones obtained by applying each transform.



- If each mapping w_1, w_2, w_3 are contractive, the resulting mapping is also contractive.

Proof:



$$h(B \cup C, D \cup E) \leq h(B, D) \vee h(C, E)$$

If we take two mappings w_1 and w_2, and consider the union mapping $W(B) = w_1(B) \cup w_2(B)$, then

$$\begin{aligned} h(W(B), W(C)) &= h(w_1(B) \cup w_2(B), w_1(C) \cup w_2(C)) \\ &\leq h(w_1(B), w_1(C)) \vee h(w_2(B), w_2(C)) \\ &\leq s_1 h(B, C) \vee s_2 h(B, C) \leq s \cdot h(B, C) \end{aligned}$$

where $s = \max\{s_1, s_2\}$

To summarize:

- If the affine transforms w_1, w_2, w_3 etc. are contractive, the union mapping is also contractive.
- These are contractive if the eigenvalues of the

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

matrix are less than unity.

- If the mapping is contractive, Banach's contraction mapping theorem ensures that on repeated iteration, it must converge on a limit point—which is also an image.

Example: Consider the Iterated Function System (IFS)

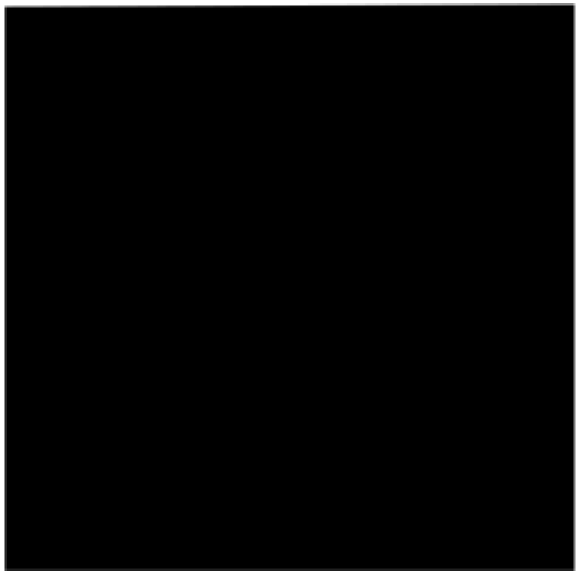
$$w_1 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

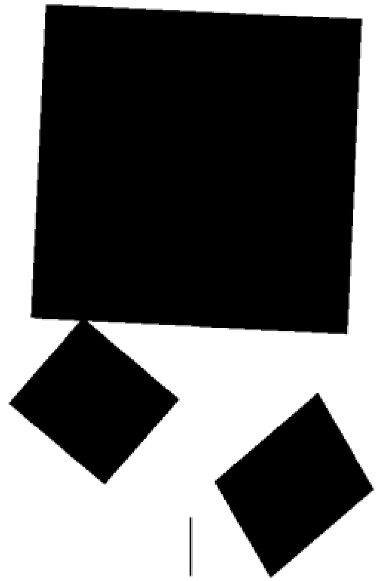
$$w_2 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$

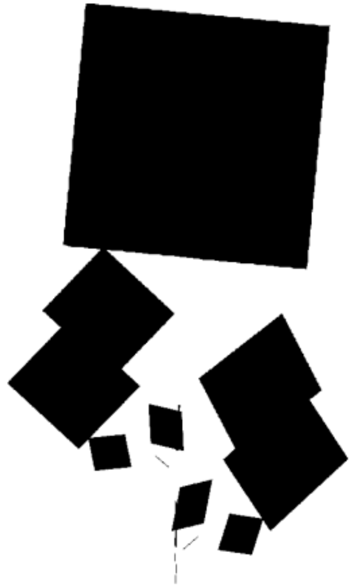
$$w_3 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$

$$w_4 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}$$

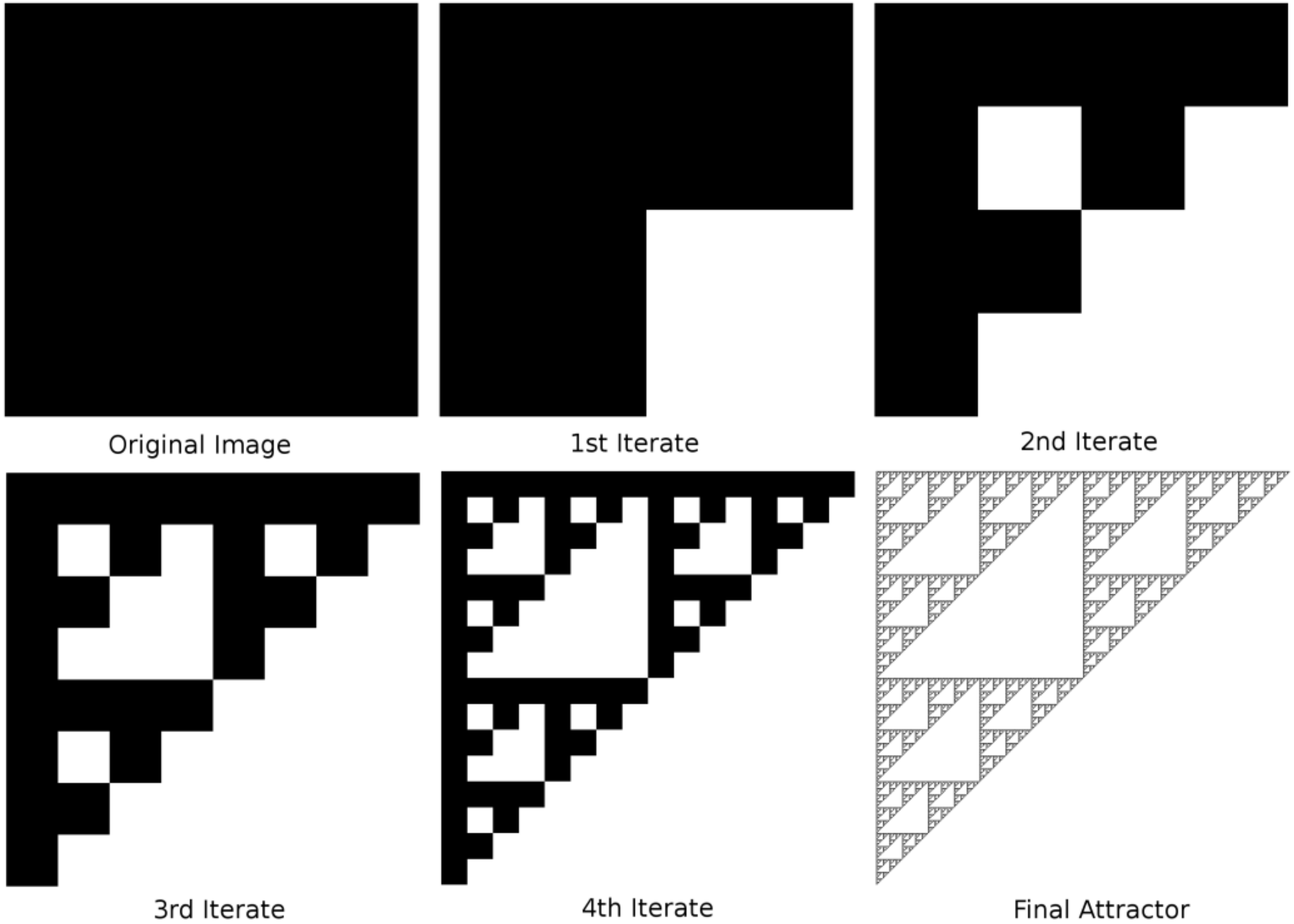
When this mapping is applied repeatedly on a square, it converges to $\cdots$







Similarly, for the Sierpinski triange,



How can we construct the iterated function system?

Imagine that you have an image in mind, L. Suppose you construct an IFS $W : \{w_1, w_2 \cdots w_n\}$ with contractivity factor $0 \leq s < 1$ such that

$$h(L, W(L)) = h\left(L, \cup_{n=1}^{n} w_n(L)\right) \leq \epsilon$$

Now consider the step in the Banach's contraction mapping theorem:

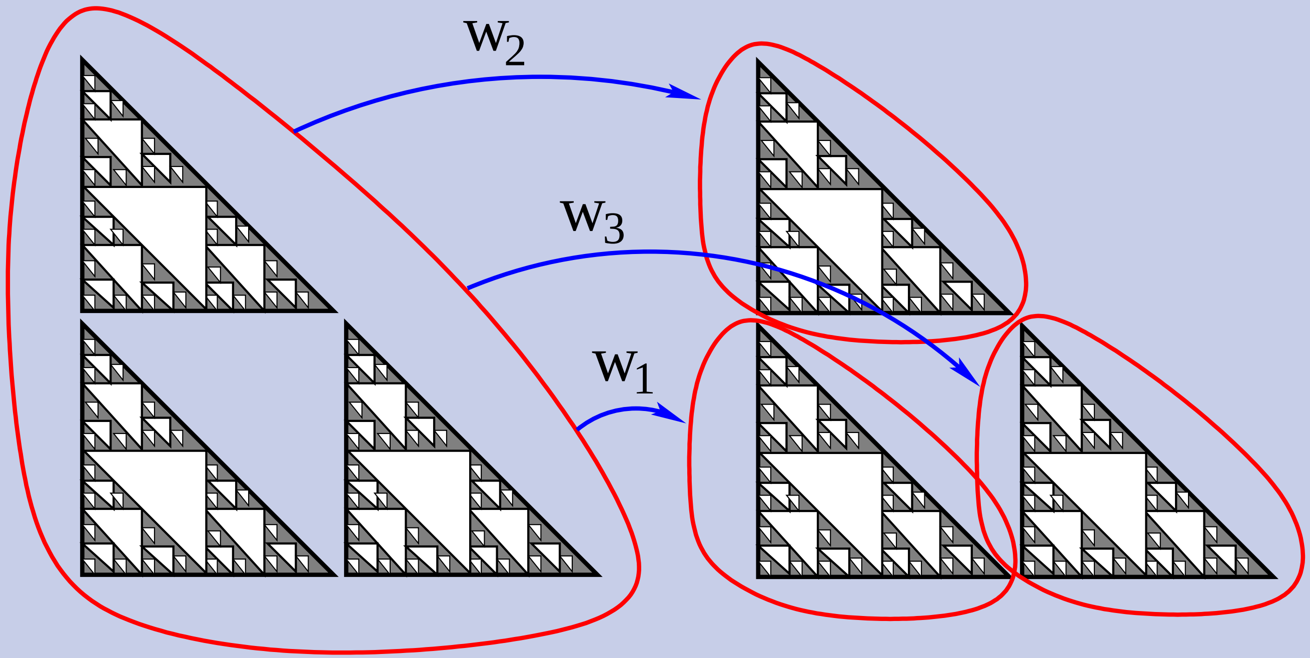
$$d(x_m, x_n) \quad \leq \quad d(x_0, x_1)\frac{s^m}{1-s}$$

Let $m = 0$ and $n = \infty$. Therefore x_m is the object L, and x_n is the attractor of the IFS, say A. Then we get

$$h(L, A) \leq \frac{h\left(L, W(L)\right)}{1-s} \leq \frac{\epsilon}{1-s}$$

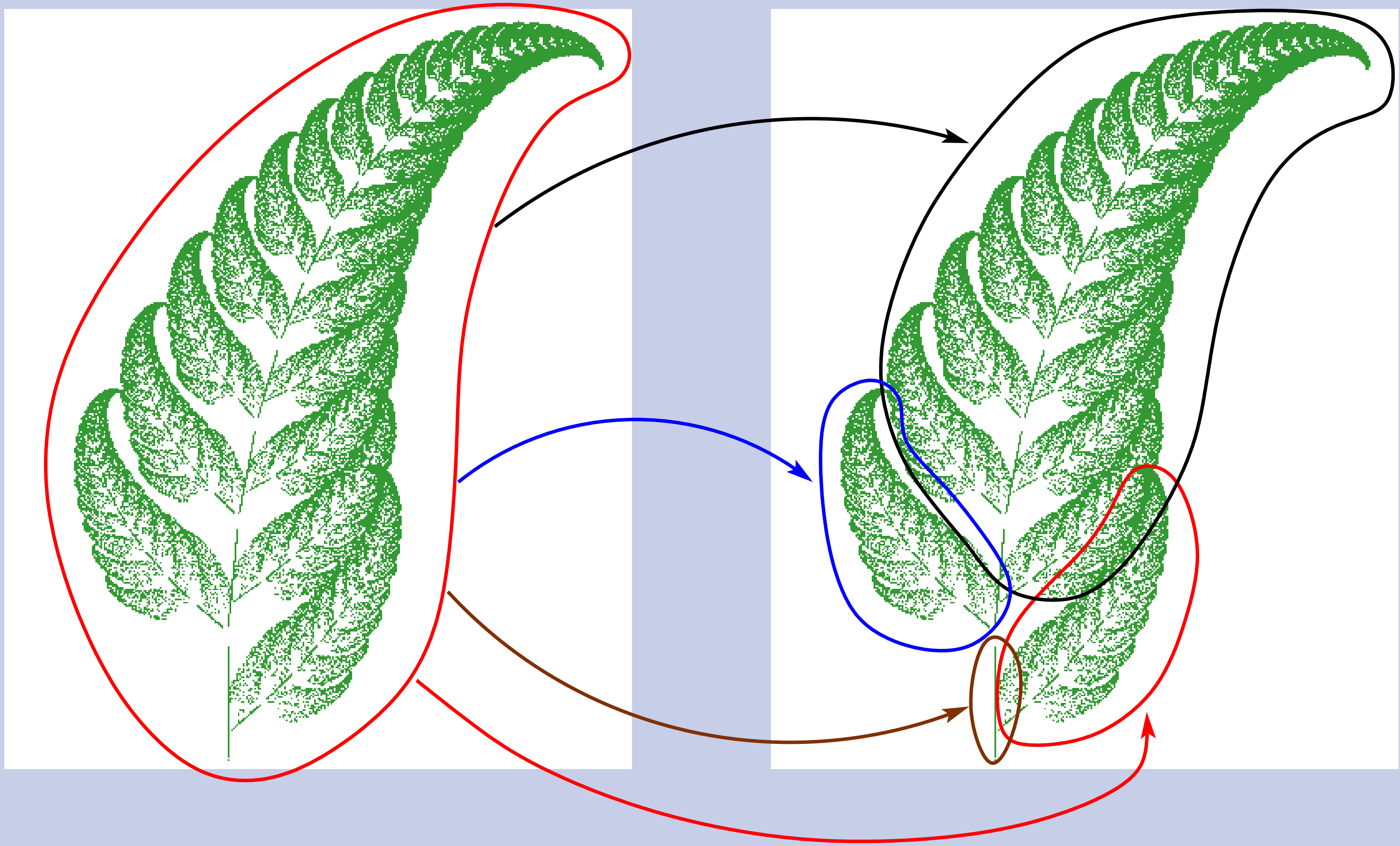
(Collage Theorem)

Using the Collage Theorem to design IFS.

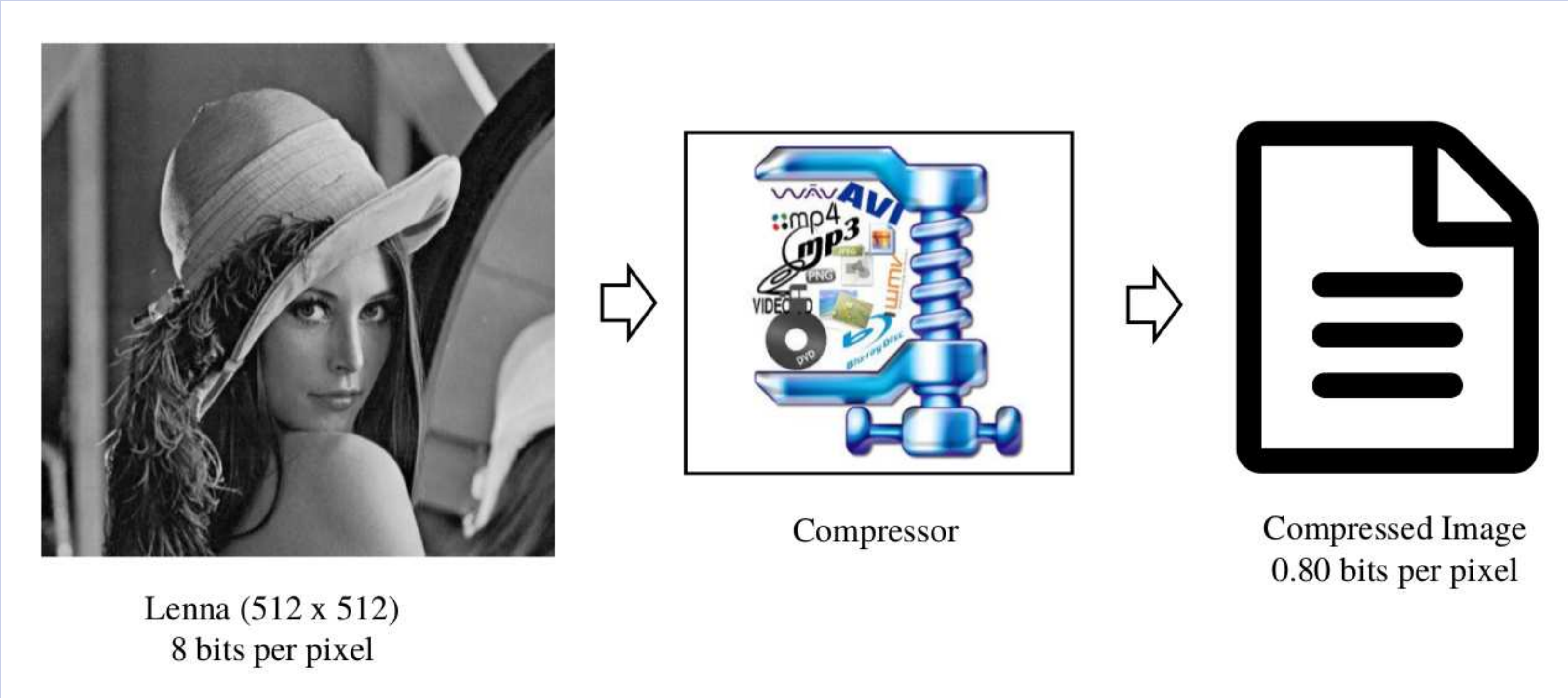


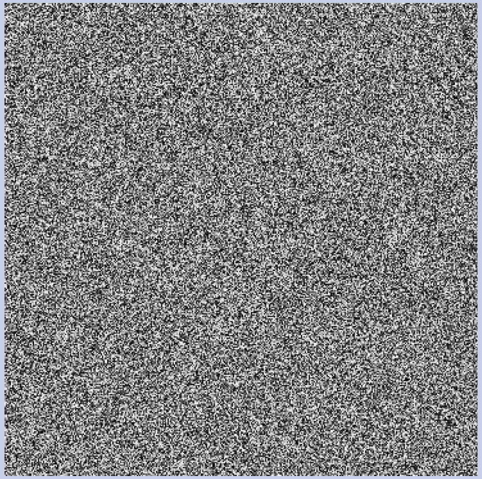
	a	b	c	d	e	f
w_1	0.5	0	0	0.5	0	0
w_2	0.5	0	0	0.5	0	50
w_3	0.5	0	0	0.5	50	0

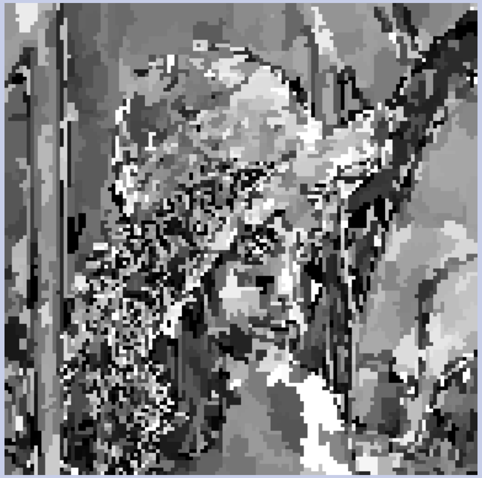
Using the Collage Theorem to design IFS.



Can we apply the idea to codify images?

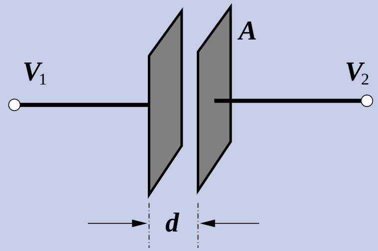






Application 2:

Super Capacitors



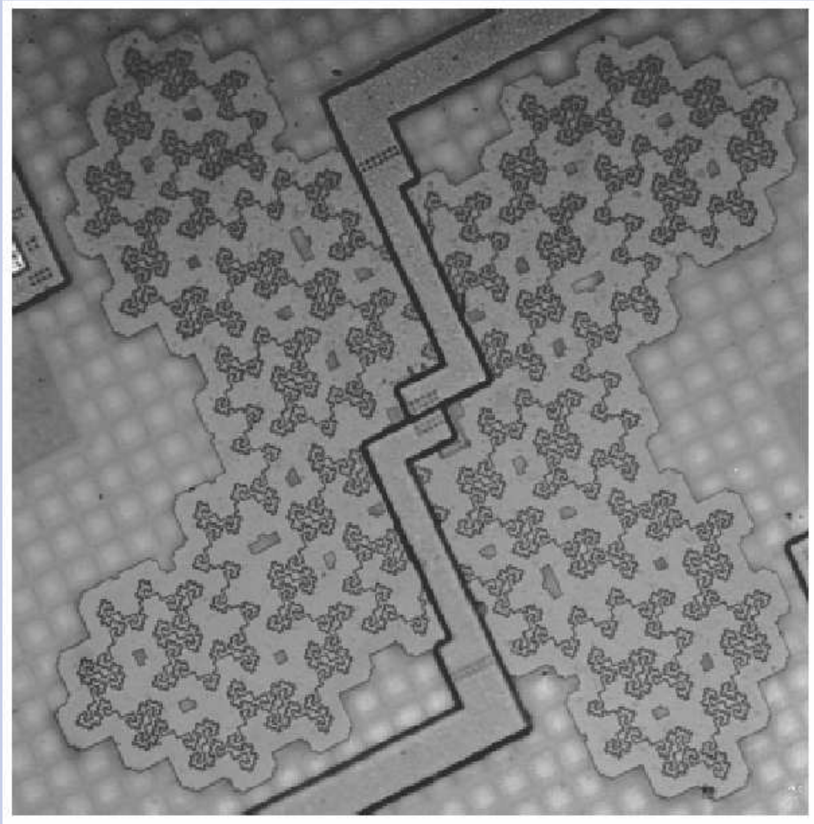
$$C = \epsilon_0 \epsilon_r \frac{A}{d}$$

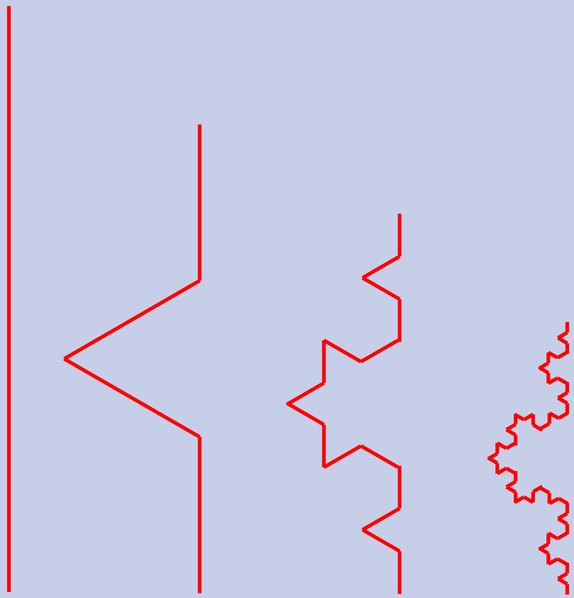
Energy stored $= \frac{1}{2} C (V_2 - V_1)^2$

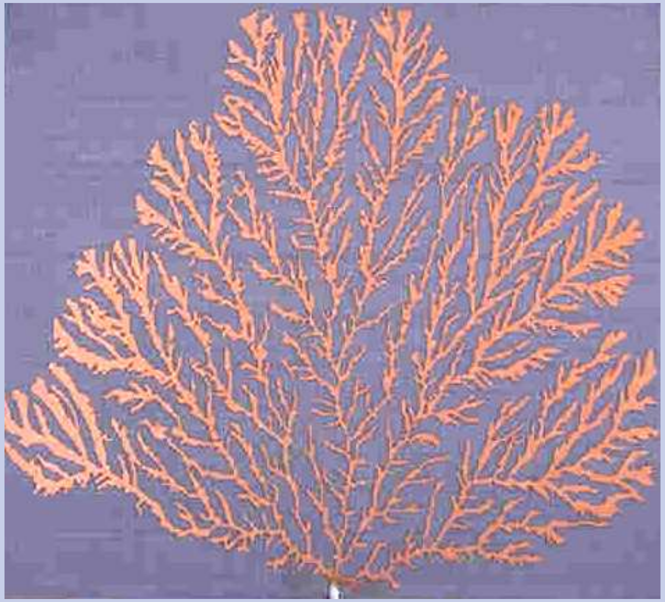
where $\epsilon_0 = 8.85 \times 10^{-12}$ F/m.

Solution: increase the area by creating fractal surfaces. 30-60 F capacitors have been developed with internal resistance less than 20mΩ.

Application 3: Capacitor integrated in ICs





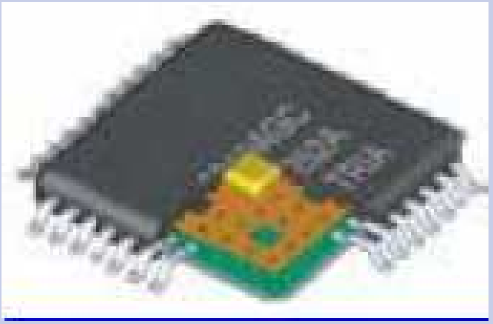


Application 4: Fractal antennas

- ➜ Small size
- ➜ Increased input impedence
- ➜ Decreased resonant frequency
- ➜ Multiband/wideband
- ➜ Integrated antenna in VLSI chip

Some early developments in fractal antennas

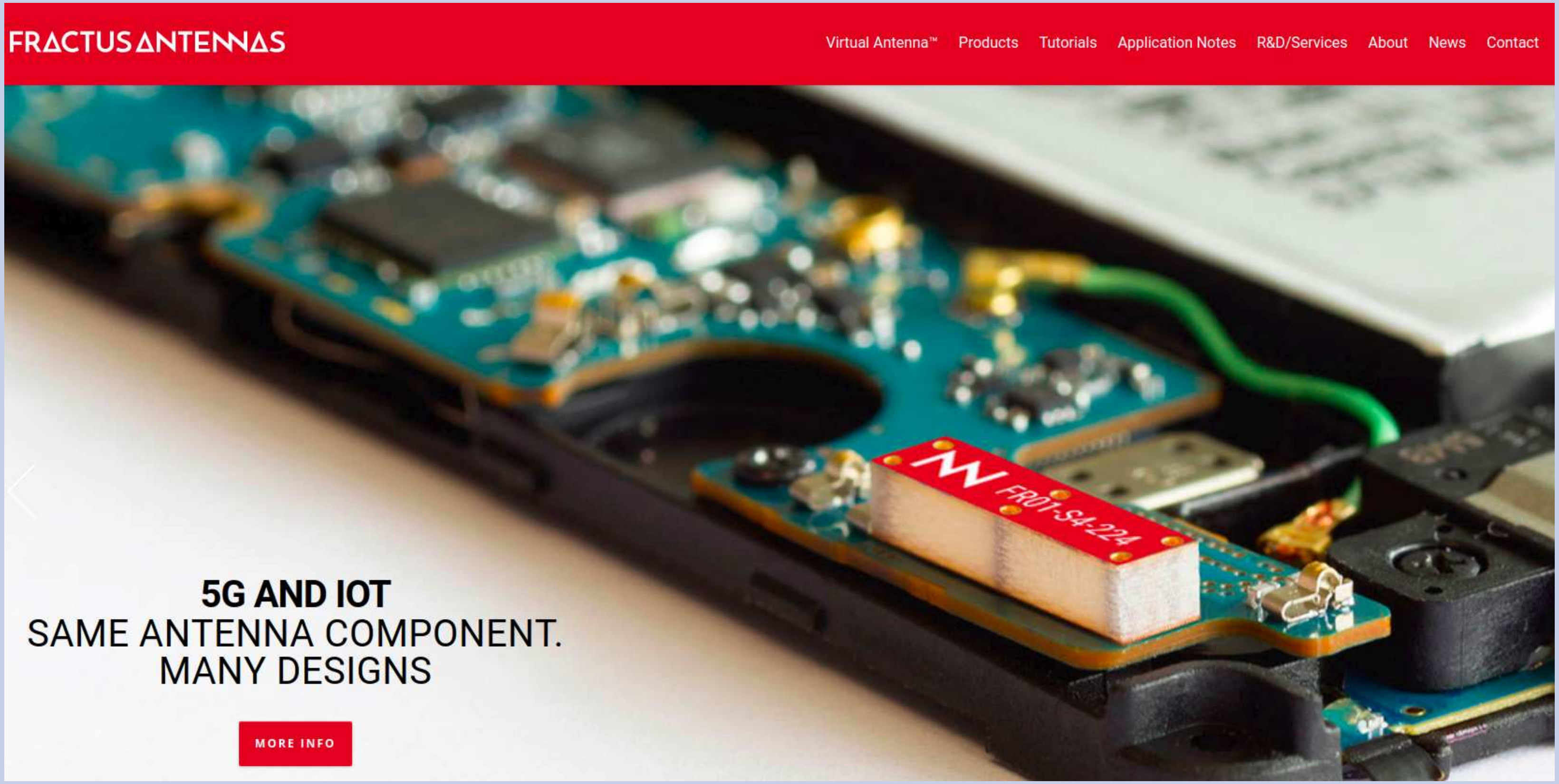
Antenna packaged in VLSI chip



Visualization of antenna (the brown layer) integrated on a package substrate.

The chip integrated in a blue-tooth adapter

And when it became industrial product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