# Breaking the Riemann Integral

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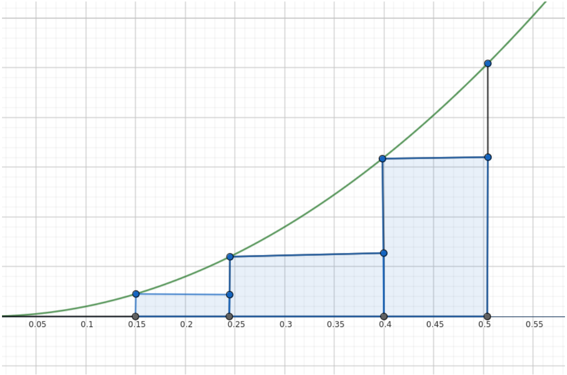
# The Riemann Integral




Given a bounded function $f : [a, b] \to \mathbb{R}$ we can define the 'signed area' under it by approximating for a partition $(x_i)_{i=0}^{n}$ of $[a, b]$:

$$L(f, P) = \sum_i m_i(x_i - x_{i-1})$$

$$U(f, P) = \sum_i M_i(x_i - x_{i-1})$$

# Riemann Integral-Contd.

If there is a number such that $I = \inf_P\{U(f,P)\} = \sup_P\{L(f,P)\}$ then $f$ is Riemann Integrable and $\int_a^b f = I$

# Metric Spaces

Suitable generalisation of spaces like $\mathbb{R}$ where we have a notion of distance/ a function $d : X \times X \to [0, \infty)$ such that:

- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x), \forall x, y \in X$
- $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X$

$(x_n)_{n=1}^{\infty}$ converges to $x \iff$ Given any $\epsilon > 0$, there exists $N$ such that $d(x_n, x) < \epsilon$ when $n \geq N$

# Completeness

A Cauchy Sequence is a sequence $(x_n)_{n=1}^{\infty}$ such that given any $\epsilon > 0$, there exists $N$ such that $d(x_n, x_m) < \epsilon, \forall n, m \geq N$. $X$ is called complete if all Cauchy sequences in $X$ converge to some element.

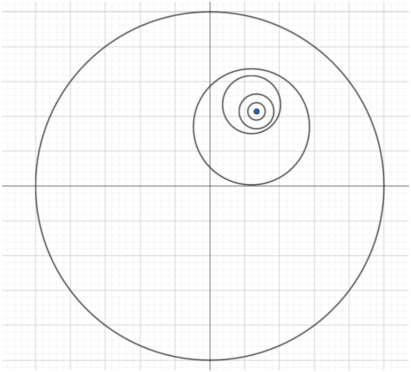


Figure: The space has no 'holes'

# Metrics on Spaces of Integrable Functions

- $L^1$ and $L^2$ form natural metrics on function spaces(after proper identification)
- It is known that under Riemann integration these spaces will not be complete. But how bad can it get?(Can we get away with Improper Integrals?)

# Well...

"If this bothers you, you should have been a math major.."
DJ Griffiths, Introduction to Quantum Mechanics

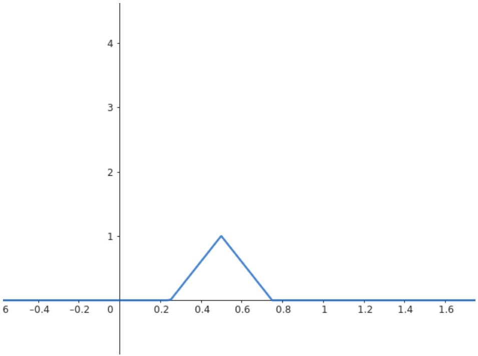
# Relevance of Completeness

- Riesz-Fischer Theorem: $L^2([0,1]) \longleftrightarrow l^2(\mathbb{Z})$ are isometric (there is a distance preserving bijective map)
- The Fourier Transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isometry
- Weak Solutions of PDEs and convergence of good 'approximations'

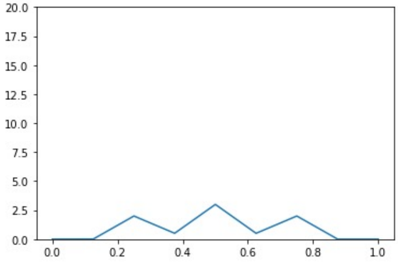
# Main idea behind the construction - Heuristics

- The fundamental definition itself requires boundedness 'somewhere atleast'
- If we create a sequence of functions that 'blows up' in a 'dense' set, it might not remain Riemann Integrable anywhere- but also has to break **nearby** points- continuity
- 'Total size' has to remain conserved- Cauchyness

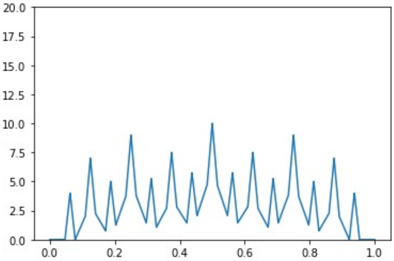
# Step 1



# Step 2



# And on...



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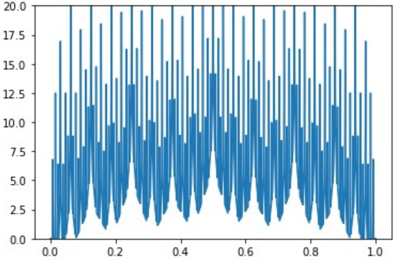

# Making it Precise

- The base of the triangles has been selected in such a way that it becomes smaller and smaller- but the total area stays conserved
- At each stage $n$ , going to rational numbers of the form $m/2^n$ for $m \in \{1, 2, \ldots 2^n - 1\}$ and adding a triangle of base $\approx 1/4^n$ and heights going to $n$
- We can estimate $d(f_n, f_m)$ to be less than $\sum_{k=m+1}^{n} k/2^k$ which can be made small if we have $n, m$ large enough.

# Breaking Riemann Integration

Suppose there exists $g : [a, b] \to \mathbb{R}$ such that $[a, b] \subset [0, 1]$ such that $d(g, f_n|_{[a,b]}) \to 0$ as $n \to \infty$ and $g$ is Riemann integrable. We can show there is a subsequence which converges pointwise **almost everywhere**

But $g$ is bounded and by the construction, we have a **segment** inside $[a, b]$ such that $f_n$'s are larger than a given integer. So we get a contradiction $\square$

# References and Acknowledgements

[1] *Real and Complex Analysis*- Walter Rudin
[2] *Functional Analysis*- Peter Lax

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More detailed version in article format available upon request.