The World Of Triangular Partitions

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November 9, 2023

A Stroll

Partition

Partition

For $n, p \in \mathbb{N}$, a **partition** of n into p parts is a weakly decreasing sequence of positive integers $\lambda = \lambda_1\lambda_2\cdots\lambda_p$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_p = n$.

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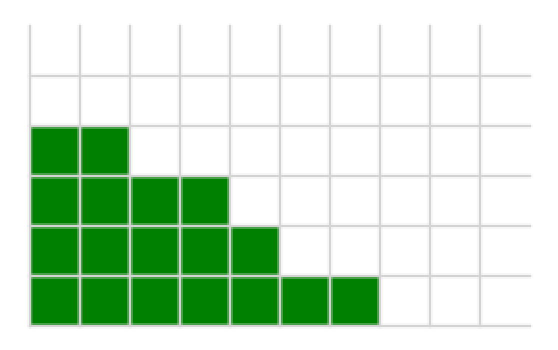

Figure: Ferrers diagram of the partition 7542

Conjugate

The conjugate of λ is denoted λ'.

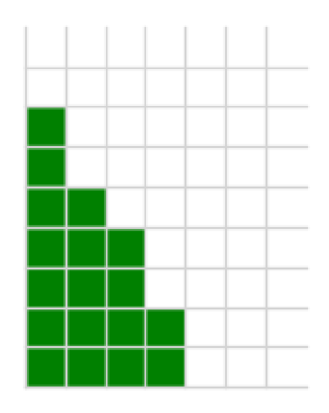

Figure: The diagram of 4433211, the conjugate of 7542

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Triangular Partition

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Given $r, s \in \mathbb{R}^+$, we define the partition τ_{rs} as follows.

$$(\tau_{rs})_i = \lfloor r - jr/s \rfloor \quad \text{for} \quad 1 \leq j \leq \lfloor s \rfloor$$

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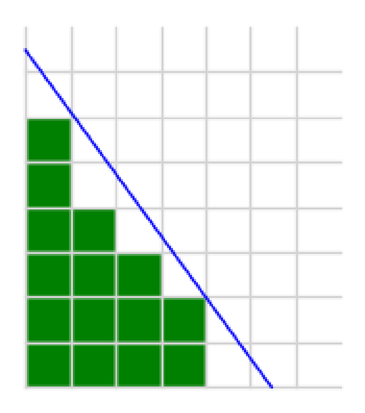

Figure: The diagram of $\tau_{rs} = 443211$ for $r = 5.5,\ s = 7.5$

A partition $\tau = \tau_1\tau_2 \cdots \tau_p$ is said to be **triangular** if there exist $r, s \in \mathbb{R}^+$ such that

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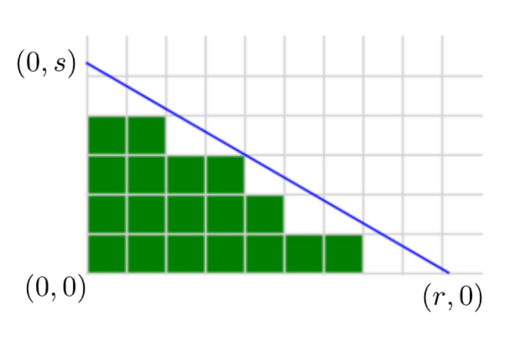

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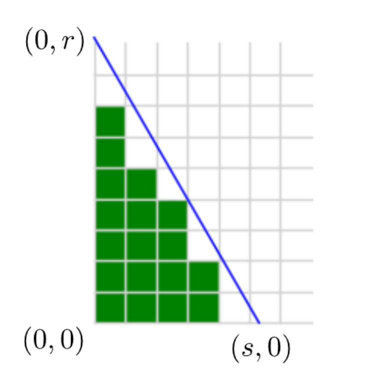

Figure: The conjugate of a triangular partition is triangular. $\tau'_{rs} = \tau_{sr}$

Triangular Partition

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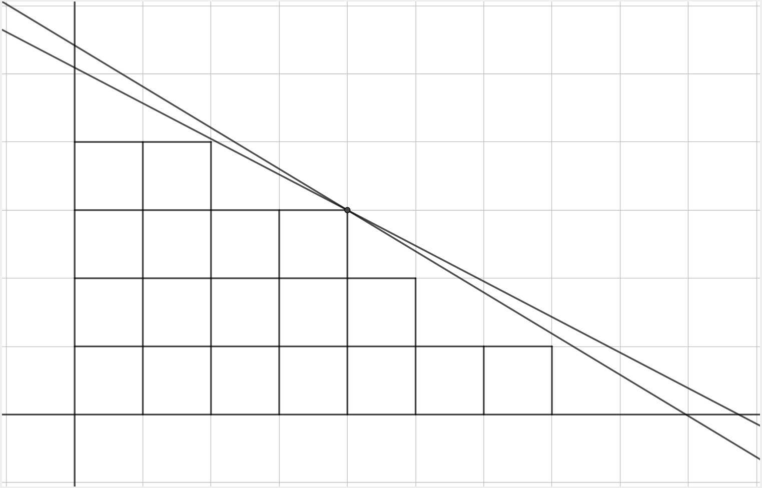

Staircase Partition

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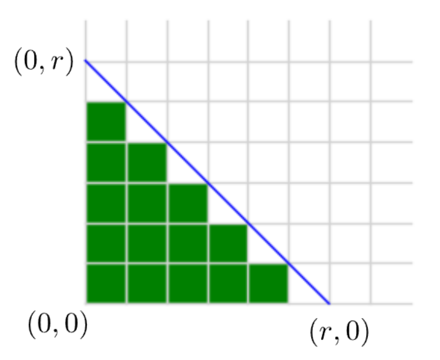

Figure: A staircase partition

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$$\tau = M(M-1)(M-2)\cdots 321 \impliedby \tau \text{ is a staircase partition}$$

$$\tau = \tau_{M+1,M+1} \impliedby \tau_j = M + 1 - j = \lfloor M + 1 - j \rfloor \impliedby$$

Addition of Partitions

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For partitions $\lambda = \lambda_1\lambda_2 \cdots \lambda_p$ and $\mu = \mu_1\mu_2 \cdots \mu_s$ with $p \geq s$, we define **addition** part-wise as follows :

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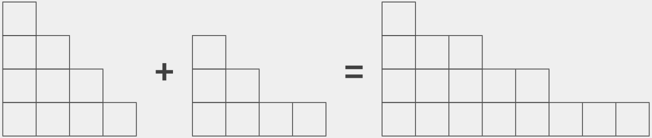

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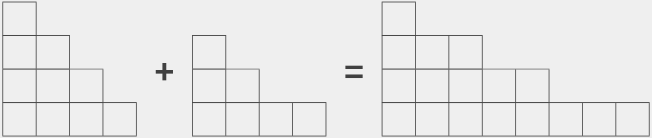
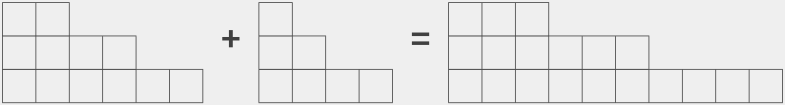


- $l(\lambda + \mu) = \max\{l(\lambda), l(\mu)\}$
- $|\lambda + \mu| = |\lambda| + |\mu|$

Addition of Triangular Partitions



$$642 + 421 = (10)63$$

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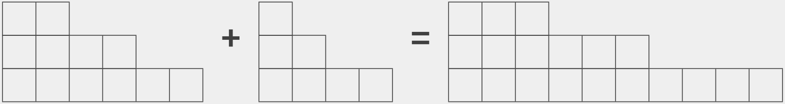

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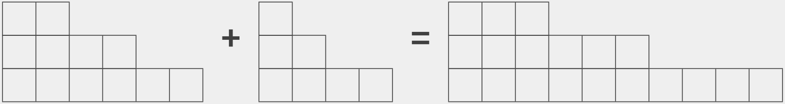
$$\textit{triangular} + \textit{triangular} = \textit{triangular}$$



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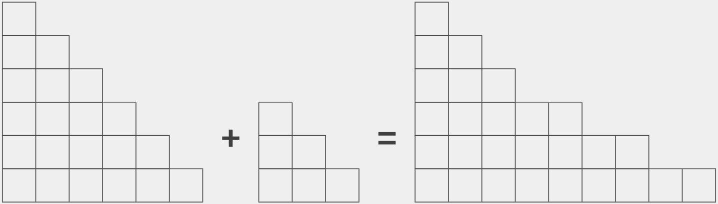

$$654321 + 321 = 975321$$

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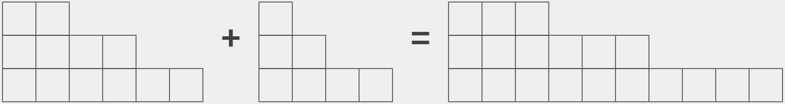

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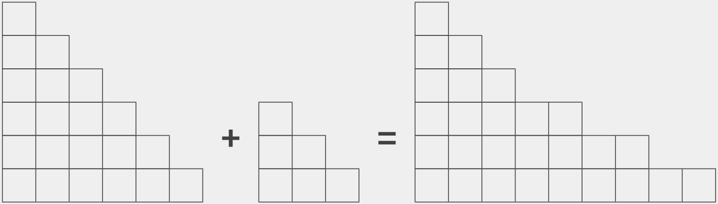

$654321 + 321 = 975321$

triangular + *triangular* = *not triangular*

The Question. For triangular partitions λ and μ, when is $\lambda + \mu$ a triangular partition?

An Encounter

Properties of Triangular Partitions and their Generalizations

Alejandro Basilio Galván Pérez-Ilzarbe

Bachelor Thesis, Universitat Politècnica de Catalunya, 2023

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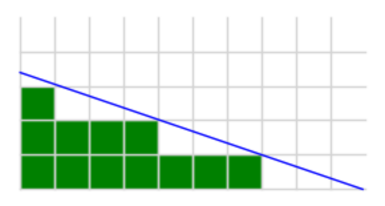

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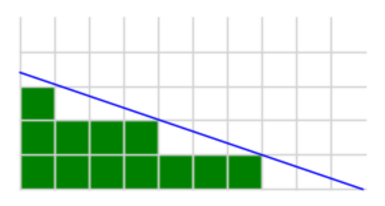

Figure: *wide*

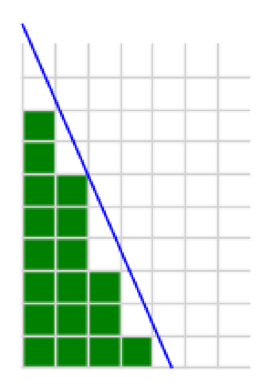

Figure: *tall*

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$$\frac{x}{r} + \frac{y}{s+\varepsilon} = 1 \text{ is also a cutting line for } \varepsilon \textit{ tiny enough}.$$

$$\implies \tau_{rs} = \tau_{r(s+\varepsilon)}$$

A Lemma

Let $p \in \mathbb{N}$ and let $\tau = \tau_1\tau_2 \cdots \tau_p$ be a triangular partition. Then,

- τ is wide if and only if there is no $i \in \{1, \ldots, p-1\}$ such that $\tau_i = \tau_{i+1}$.
- τ is tall if and only if τ' is wide.
- τ is wide and tall if and only if it is a staircase partition.

And Its Proof

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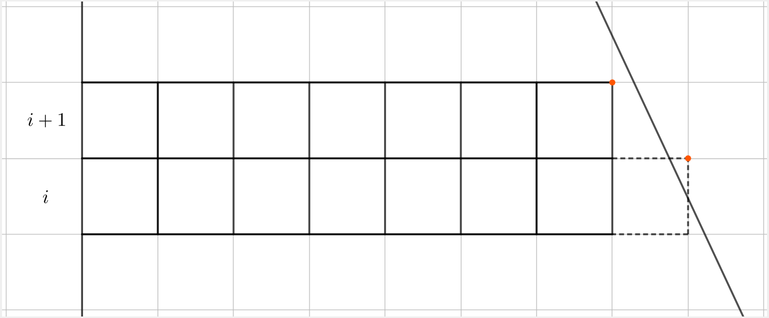

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$$\tau \text{ is wide and tall} \iff \tau = \tau_{ab} = \tau_{a'b'} \text{ with } a > b \;\&\; a' < b'$$

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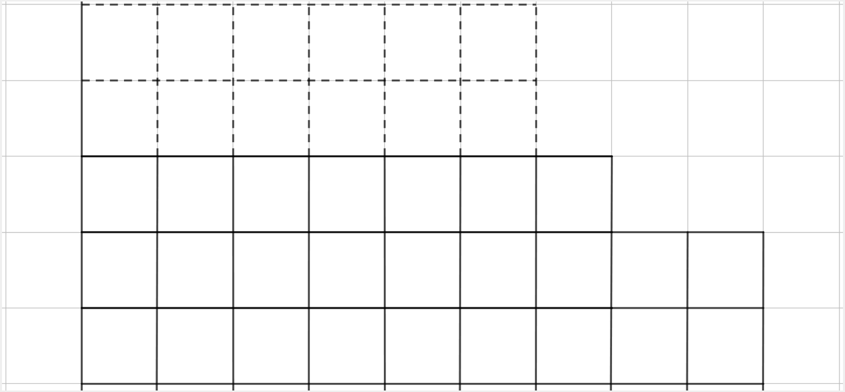

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If $u = u_1 u_2 \dots u_k,\ v = v_1 v_2 \dots v_s \in A^*$, we define ***concatenation*** as

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$$\implies A^* \text{ is a monoid, in fact the } \textit{free monoid on A}$$

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- For $b \in A$, $|w|_b =$ number of letters in w which are b

Balanced Words

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A word $w = w_1 w_2 \ldots w_k$ is said to be **balanced** if for any two factors u, v of w of the same length, $||u|_1 - |v|_1| \leq 1$.

Quasi-Balanced Words

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$$111\ldots111 \sim 000\ldots000$$

Let $\tau = \tau_1 \tau_2 \cdots \tau_p$ be a triangular partition.

The Set of Steps

Let $\tau = \tau_1\tau_2 \cdots \tau_p$ be a triangular partition.

If $p \geq 2$, $\quad S_\tau = \{\tau_{p-1} - \tau_p, \tau_{p-2} - \tau_{p-1}, \ldots \tau_2 - \tau_3, \tau_1 - \tau_2\}$

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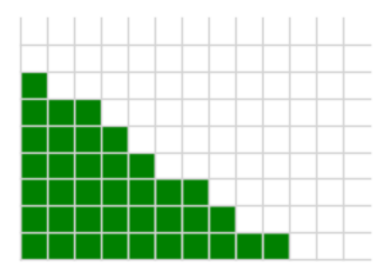

Figure: $\tau = (10)875431$

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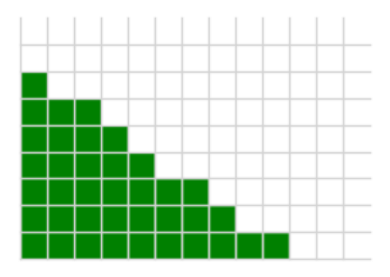

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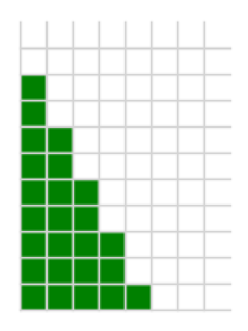
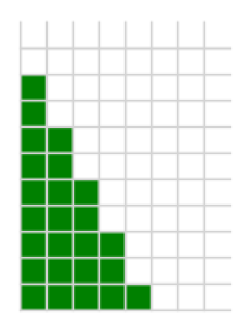


Figure: $\lambda = 544332211$

$S_\lambda = \{0, 1\}$



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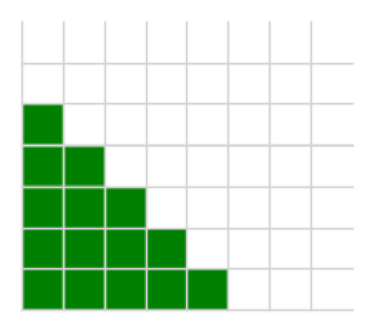

Figure: $\mu = 54321$

Another Lemma

Let $\tau = \tau_1 \tau_2 \cdots \tau_p$ be a wide triangular partition. Then, there exists $m \in \mathbb{N}$ such that $\tau_p \leq m + 1$ and either $S_\tau = \{m\}$ or $S_\tau = \{m, m+1\}$.

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Proof. If $p = 1, \ \ S_\tau = \{\tau_1\} \implies m = \tau_1$

$$\tau = \tau_{rs} \implies \tau_i = \lfloor r - ir/s \rfloor$$

$$x - 1 < \lfloor x \rfloor \leq x \quad \mathit{for}\ x \in \mathbb{R}$$

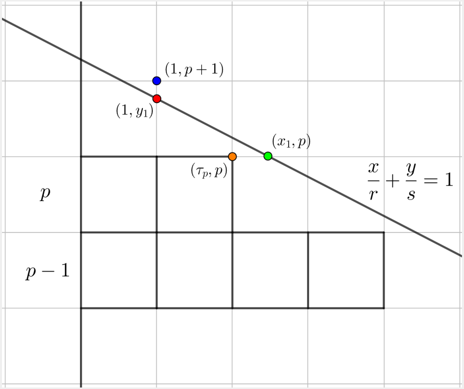
$$\tau_i - \tau_{i+1} = \lfloor r - ir/s \rfloor - \lfloor r - (i+1)r/s \rfloor$$

$$r - ir/s - 1 - (r - (i+1)r/s) < \tau_i - \tau_{i+1} < r - ir/s - (r - (i+1)r/s - 1)$$

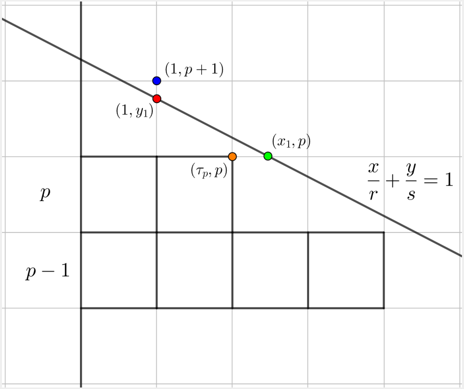
$$\implies r/s - 1 < \tau_i - \tau_{i+1} < r/s + 1$$

$$\implies |S_\tau| \leq 2$$

Another Lemma



Another Lemma



$$y_1 = (1 - 1/r)s < p + 1 \implies 1 - 1/r < p/s + 1/s \implies 1 - p/s < 1/r + 1/s$$

$$\tau_p \leq x_1 = (1 - p/s)r \implies \tau_p \leq (1/r + 1/s)r = 1 + r/s < m + 2$$

$$\implies \tau_p \leq m + 1$$

The Lemma Let $\tau = \tau_1\tau_2 \cdots \tau_p$ be a wide triangular partition. Then, there exists $m \in \mathbb{N}$ such that $\tau_p \leq m + 1$ and either $S_\tau = \{m\}$ or $S_\tau = \{m, m + 1\}$.

The Lemma Let $\tau = \tau_1\tau_2 \cdots \tau_p$ be a wide triangular partition. Then, there exists $m \in \mathbb{N}$ such that $\tau_p \leq m + 1$ and either $S_\tau = \{m\}$ or $S_\tau = \{m, m + 1\}$.

Its Twin Let $\tau = \tau_1\tau_2 \ldots \tau_p$ be a tall triangular partition which is not a staircase. Then, $\tau_p = 1$ and either $S_\tau = \{0\}$ or $S_\tau = \{0, 1\}$.

Let $\tau = \tau_1\tau_2\cdots\tau_p$ be a triangular partition.

The Characterization

Let $\tau = \tau_1\tau_2\cdots\tau_p$ be a triangular partition.

- $\chi_1(\tau) = \tau_p$

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- $\chi_1(\tau) = \tau_p$
- $\chi_2(\tau) = \begin{cases} m & \text{if } S_\tau = \{m, m+1\} \text{ or } S_\tau = \{m\} \text{ and } \chi_1(\tau) \geq m \\ m-1 & \text{if } S_\tau = \{m\} \text{ and } \chi_1(\tau) \leq m-1 \end{cases}$

The Characterization

Let $\tau = \tau_1 \tau_2 \cdots \tau_p$ be a triangular partition.

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- $\chi_3(\tau) = w_1 w_2 \ldots w_p$

$$\textit{For } \; i \in \{2, \ldots, p\}, \;\; w_i = 1 \iff \tau_{p-i+1} - \tau_{p-i+2} = \chi_2(\tau) + 1$$

$$w_1 = 1 \iff \tau_p = \chi_2(\tau) + 1$$

$$S_\tau = \{1, 2\}$$

$$\chi_1(\tau) = 1$$

$$\chi_2(\tau) = 1$$

$$\chi_3(\tau) = 0100101$$

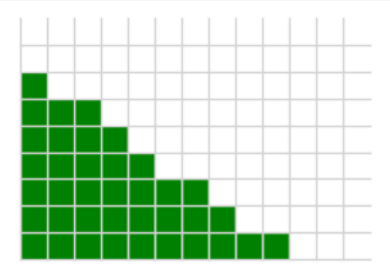

Figure: $\tau = (10)875431$

$$S_\lambda = \{0, 1\}$$

$$\chi_1(\lambda) = 1$$

$$\chi_2(\lambda) = 0$$

$$\chi_3(\lambda) = 101010101$$

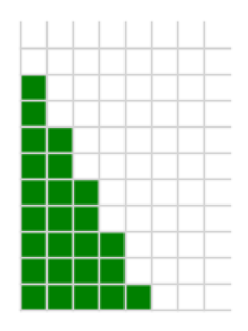

Figure: $\lambda = 544332211$

The Characterization

$$S_\mu = \{1\}$$

$$\chi_1(\mu) = 1$$

$$\chi_2(\mu) = 1$$

$$\chi_3(\mu) = 00000$$

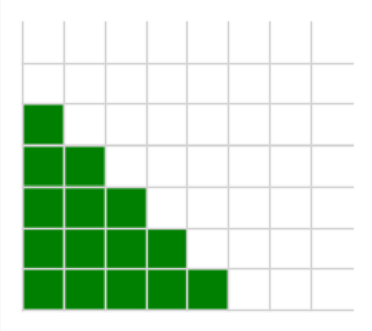

Figure: $\mu = 54321$

The map $\chi = (\chi_1, \chi_2, \chi_3)$ is a bijection between the set of wide triangular partitions and the set

$$\{(\ell_0, \ell, w) \in \mathbb{N} \times \mathbb{N} \times \mathcal{QB}_0 : \ell_0 = \ell + 1 \text{ if } w_1 = 1;\ \ell_0 = \ell \text{ if } w = 0 \ldots 0;\ \ell_0 \leq \ell \text{ otherwise}\}.$$

The map $\chi = (\chi_1, \chi_2, \chi_3)$ is a bijection between the set of wide triangular partitions and the set

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The map $\chi = (\chi_1, \chi_2, \chi_3)$ is a bijection between the set of tall triangular partitions which are not staircases and the set

$$\{(1, 0, w) : w \in \mathcal{QB}_0 \ \ \&\ \ w_1 = 1\}.$$

An Effort

The Question. For triangular partitions λ and μ, when is $\lambda + \mu$ a triangular partition?

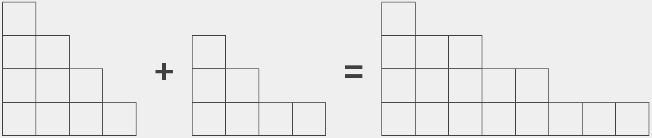
The Method

The Method

$$(\chi_1(\lambda+\mu), \chi_2(\lambda+\mu), \chi_3(\lambda+\mu)) \text{ ?}$$

$$l(\lambda) = l(\mu) \text{ or } l(\lambda) > l(\mu)$$

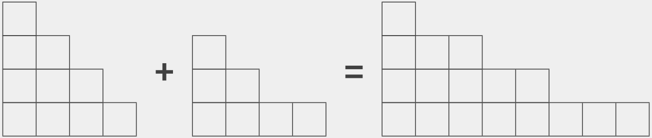
$$s_{\lambda+\mu} \text{ ?}$$



$$4321 + 421 = 8531$$

For $1 \le i \le l(\mu) - 1$,

$$\begin{aligned}(\lambda+\mu)_i - (\lambda+\mu)_{i+1} &= \lambda_i + \mu_i - \lambda_{i+1} - \mu_{i+1} \\ &= (\lambda_i - \lambda_{i+1}) + (\mu_i - \mu_{i+1})\end{aligned}$$



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$$\textit{For} \ \ 1 \leq i \leq l(\mu) - 1,$$

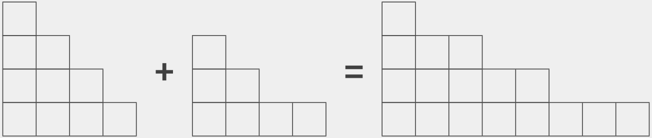
$$\begin{aligned}(\lambda + \mu)_i - (\lambda + \mu)_{i+1} &= \lambda_i + \mu_i - \lambda_{i+1} - \mu_{i+1} \\ &= (\lambda_i - \lambda_{i+1}) + (\mu_i - \mu_{i+1})\end{aligned}$$

$$\begin{aligned}(\lambda + \mu)_{l(\mu)} - (\lambda + \mu)_{l(\mu)+1} &= (\lambda_{l(\mu)} - \lambda_{l(\mu)+1}) + \mu_{l(\mu)} \\ &= (\lambda_{l(\mu)} - \lambda_{l(\mu)+1}) +_1 (\mu)\end{aligned}$$

$$\textit{For} \ \ l(\mu) + 1 \leq j \leq l(\lambda) - 1,$$

$$(\lambda + \mu)_j - (\lambda + \mu)_{j+1} = \lambda_j - \lambda_{j+1}$$

The Method



$$4321 + 421 = 8531$$

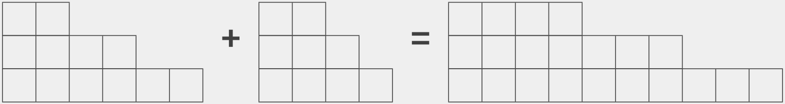
$$\begin{aligned} \chi_1(\lambda + \mu) &= (\lambda + \mu)_{l(\lambda)} = \lambda_{l(\lambda)} + \mu_{l(\lambda)} \\ &= \begin{cases} \chi_1(\lambda) + \chi_1(\mu) & \text{if } l(\lambda) = l(\mu) \\ \chi_1(\lambda) & \text{if } l(\lambda) > l(\mu) \end{cases} \end{aligned}$$

Demonstration I

Demonstration I

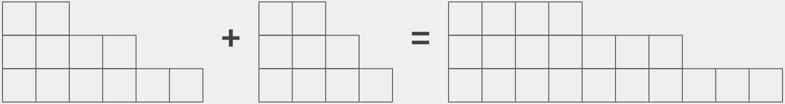
- Let λ be a wide triangular partition and μ be a partition with $l(\lambda) \geq l(\mu)$. If $\lambda + \mu$ is triangular, then $\lambda + \mu$ is a wide triangular partition.

  - $l(\lambda) = l(\mu),\ S_\lambda = \{m\},\ ,S_\mu = \{n\},\ \chi_1(\lambda) \geq m,\ \chi_1(\mu) \geq n$



$$642 + 432 = (10)74$$

- $l(\lambda) = l(\mu),\ S_\lambda = \{m\},\ S_\mu = \{n\},\ \chi_1(\lambda) \geq m,\ \chi_1(\mu) \geq n$



$$642 + 432 = (10)74$$

$$S_{\lambda+\mu} = \{m+n\},\ \chi_1(\lambda+\mu) = \chi_1(\lambda) + \chi_1(\mu) \geq m+n$$

$$\chi_2(\lambda+\mu) = m+n$$

$$\chi_3(\lambda+\mu) = \begin{cases} 1000\ldots000 & \text{if } \chi_1(\lambda) = m \ \&\ \chi_1(\mu) = n+1 \text{ or} \\ & \chi_1(\lambda) = m+1 \ \&\ \chi_1(\mu) = n \\ 0000\ldots000 & \text{if } \chi_1(\lambda) = m \ \&\ \chi_1(\mu) = n \end{cases}$$

Demonstration I

- $l(\lambda) = l(\mu),\ S_\lambda = \{m\},\ S_\mu = \{n\},\ \chi_1(\lambda) \geq m,\ \chi_1(\mu) \geq n$

$$S_{\lambda+\mu} = \{m+n\},\ \chi_1(\lambda+\mu) = \chi_1(\lambda) + \chi_1(\mu) \geq m+n$$

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A Bad Case

$\chi_1(\lambda) = m+1 \ \&\ \chi_1(\mu) = n+1$ would imply that $\chi_1(\lambda+\mu) = m+n+2 \not\leq m+n+1$.

Demonstration I

- $l(\lambda) = l(\mu),\ S_\lambda = \{m\},\ S_\mu = \{n\},\ \chi_1(\lambda) \geq m,\ \chi_1(\mu) \geq n$

- $(m + n + 1,\ m + n,\ 1000\ldots000)$
- $(m + n,\ m + n,\ 0000\ldots000)$

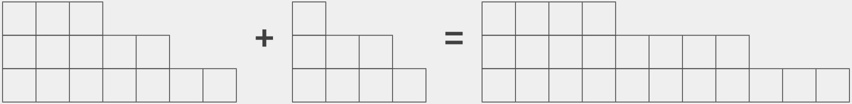
$\bullet\ l(\lambda) = l(\mu),\ S_\lambda = \{m\},\ S_\mu = \{n\},\ \chi_1(\lambda) \geq m,\ \chi_1(\mu) \geq n$

- $(m+n+1,\ m+n,\ 1000\ldots000)$
- $(m+n,\ m+n,\ 0000\ldots000)$

$\{(\ell_0, \ell, w) \in \mathbb{N} \times \mathbb{N} \times \mathcal{QB}_0 : \ell_0 = \ell + 1 \text{ if } w_1 = 1;\ \ell_0 = \ell \text{ if } w = 0\ldots0;\ \ell_0 \leq \ell \text{ otherwise}\}.$

Demonstration II

- $l(\lambda) = l(\mu),\ S_\lambda = \{m\},\ S_\mu = \{n, n+1\},\ \chi_1(\lambda) \geq m$



$$753 + 431 = (11)84$$

$$S_{\lambda+\mu} = \{m+n, m+n+1\},\ \chi_1(\lambda+\mu) = \chi_1(\lambda) + \chi_1(\mu) \geq m + \chi_1(\mu)$$

$$\chi_2(\lambda+\mu) = m+n$$

$$\chi_3(\lambda+\mu) = \begin{cases} \chi_3(\mu) & \text{if } \chi_1(\lambda) = m \\ 1 * \bullet\, \chi_3(\mu) & \text{if } \chi_1(\lambda) = m+1 \ \ \&\ \ \chi_1(\mu) = n \\ \chi_3(\mu) & \text{if } \chi_1(\lambda) = m+1 \ \ \&\ \ \chi_1(\mu) \leq n-1 \end{cases}$$

Demonstration II

- $l(\lambda) = l(\mu),\ S_\lambda = \{m\},\ S_\mu = \{n, n+1\},\ \chi_1(\lambda) \geq m$

$$S_{\lambda+\mu} = \{m+n, m+n+1\},\ \chi_1(\lambda+\mu) = \chi_1(\lambda) + \chi_1(\mu) \geq m + \chi_1(\mu)$$

$$\chi_2(\lambda+\mu) = m+n$$

$$\chi_3(\lambda+\mu) = \begin{cases} \chi_3(\mu) & \text{if } \chi_1(\lambda) = m \\ 1 * \bullet \chi_3(\mu) & \text{if } \chi_1(\lambda) = m+1\ \ \&\ \ \chi_1(\mu) = n \\ \chi_3(\mu) & \text{if } \chi_1(\lambda) = m+1\ \ \&\ \ \chi_1(\mu) \leq n-1 \end{cases}$$

Still A Bad Case

$\chi_1(\lambda) = m+1\ \ \&\ \ \chi_1(\mu) = n+1$ would imply that
$\chi_1(\lambda+\mu) = m+n+2 \not\leq m+n+1$.

Demonstration II

$\bullet\ l(\lambda) = l(\mu),\ S_\lambda = \{m\},\ S_\mu = \{n, n+1\},\ \chi_1(\lambda) \geq m$

- $(m + \chi_1(\mu)\ , m + n\ , \chi_3(\mu))$
- $(m + n + 1\ , m + n\ , 1 * \bullet\, \chi_3(\mu))$
- $(\leq m + n\ , m + n\ , \chi_3(\mu))$

$$\bullet\; l(\lambda) = l(\mu),\ S_\lambda = \{m\},\ S_\mu = \{n, n+1\},\ \chi_1(\lambda) \geq m$$

- $(m + \chi_1(\mu)\ , m + n\ , \chi_3(\mu))$
- $(m + n + 1\ , m + n\ , 1 * \bullet \chi_3(\mu))$
- $(\leq m + n\ , m + n\ , \chi_3(\mu))$

A word $w = w_1 w_2 \ldots w_k$ is said to be **quasi-balanced** if at least one of the following holds :

- w is balanced
- $w_1 = 0$ and $w' = w_2 w_3 \ldots w_k$ is balanced

- $l(\lambda) = l(\mu),\ S_\lambda = \{m\},\ S_\mu = \{n, n+1\},\ \chi_1(\lambda) \geq m$

- $(m + \chi_1(\mu)\ , m + n\ , \chi_3(\mu))$
- $(m + n + 1\ , m + n\ , 1 * \bullet\, \chi_3(\mu))$
- $(\leq m + n\ , m + n\ , \chi_3(\mu))$

A Problem. *Can we characterize all balanced words w for which 1w is a balanced word?*

Operations on Words

- For words $u = u_1 u_2 \dots u_k$ and $v = v_1 v_2 \dots v_k$, we define the **OR addition** as follows.

$$u \bigoplus v = (u_1 + v_1)(u_2 + v_2) \dots (u_k + v_k)$$

In particular, we have that $1 \bigoplus 0 = 0 \bigoplus 1 = 1 \bigoplus 1 = 1$ and $0 \bigoplus 0 = 0$.

- For words $u = u_1u_2 \ldots u_k$ and $v = v_1v_2 \ldots v_k$, we define the **OR addition** as follows.

$$u \bigoplus v = (u_1 + v_1)(u_2 + v_2) \ldots (u_k + v_k)$$

In particular, we have that $1 \bigoplus 0 = 0 \bigoplus 1 = 1 \bigoplus 1 = 1$ and $0 \bigoplus 0 = 0$.

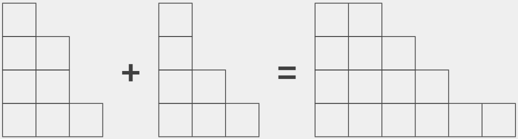
- For words $u = u_1u_2 \ldots u_k$ and $v = v_1v_2 \ldots v_k$, we define the **AND addition** as follows.

$$u \boxplus v = (u_1v_1)(u_2v_2) \ldots (u_kv_k)$$

In particular, we have that $1 \boxplus 0 = 0 \boxplus 1 = 0 \boxplus 0 = 0$ and $1 \boxplus 1 = 1$.

Demonstration III

- $l(\lambda) = l(\mu),\ S_\lambda = \{0, 1\},\ S_\mu = \{0, 1\},\ 0 \not\leftrightarrow 0,\ 1 \leftrightarrow 1$



$$3221 + 3211 = 6432$$

$$S_{\lambda+\mu} = \{1, 2\},\ \chi_1(\lambda + \mu) = \chi_1(\lambda) + \chi_1(\mu) = 2$$

$$\chi_2(\lambda + \mu) = 1$$

$$\chi_3(\lambda + \mu) = \chi_3(\lambda) \boxplus \chi_3(\mu)$$

Another Problem. *Can we characterize all balanced words u and v with* $u_1 = v_1 = 1$ *such that* $u \boxplus v$ *is a balanced word?*

Katastrophē

Let λ and μ be wide triangular partitions with $l(\lambda) \geq l(\mu)$, $S_\lambda = \{m\}$ and $S_\mu = \{n\}$ for $m, n \in \mathbb{N}$. Then, $\lambda + \mu$ is a triangular partition unless one of the following holds.

- $l(\lambda) = l(\mu)$, $\chi_1(\lambda) = m + 1$ & $\chi_1(\mu) = n + 1$
- $l(\mu) = 1$, $l(\lambda) - l(\mu) \geq 2$ & $n \geq 2$
- $l(\mu) \geq 2$, $l(\lambda) - l(\mu) = 1$ & $\chi_1(\mu) \leq n - 2$
- $l(\mu) \geq 2$, $l(\lambda) - l(\mu) = 2$ & $n \geq 2$
- $l(\mu) \geq 2$, $l(\lambda) - l(\mu) = 2$, $n = 1$ & $\chi_1(\mu) = 2$
- $l(\mu) \geq 2$ & $l(\lambda) - l(\mu) \geq 3$

$m = n = 1,\ \chi_1(\lambda) = \chi_1(\mu) = 1 \implies \lambda\ \&\ \mu$ are staircase partitions

Let λ and μ be staircase partitions with $l(\lambda) \geq l(\mu)$. Then, $\lambda + \mu$ is a triangular partition unless $l(\mu) \geq 2\ \&\ l(\lambda) - l(\mu) \geq 3$.

Open Questions Side A

- *Can we characterize all balanced words w for which 1w is a balanced word?*
- *Can we characterize all pairs of quasi-balanced words u and v for which* $u \oplus v$ *is a quasi-balanced word?*
- *Can we characterize all pairs of balanced words u and v for which* $u \oplus v$ *is a balanced word?*
- *Can we characterize all pairs of quasi-balanced words u and v for which* $u \boxplus v$ *is a quasi-balanced word?*

- *Can we characterize all pairs of balanced words u and v with* $u_1 = v_1 = 1$ *such that* $u \boxplus v$ *is a balanced word?*
- *Can we characterize all pairs of a quasi-balanced word u and a balanced word v with* $u_1 = 0$ *and* $v_1 = 1$ *for which* $u \oplus v$ *is a balanced word?*
- *Can we characterize all pairs of a quasi-balanced word u and a balanced word v with* $v_1 = 1$ *for which* $u \boxplus v$ *is a quasi-balanced word?*
- *Can we characterize all balanced words w with* $|w| \geq 2$ *for which* $0w$ *is a balanced word?*

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Merci et bonne soirée