The Nature of the Quantum State

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Observing a Century of Quantum Mechanics
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I asked him whether he felt elated or happy or transported when he discovered what is called the Bose statistics. 'If I were truly honest I can say NO. My feelings were entirely different. Let me put it this way: Suppose you had a severe headache or stomachache – and the ache suddenly stopped. That was the feeling I had. For all the previous derivations gave me ceaseless pain. If you call cessation of pain as happiness – then I can say I was happy. [...]'

S. Ramaseshan, Satyendranath Bose, A conversation with Satyendranath Bose about five decades ago – Some recollections, *Current Science*, Vol. 78, No. 5 (2000), pp. 636-638.

Is the Quantum State Real? An Extended Review of ψ-ontology Theorems

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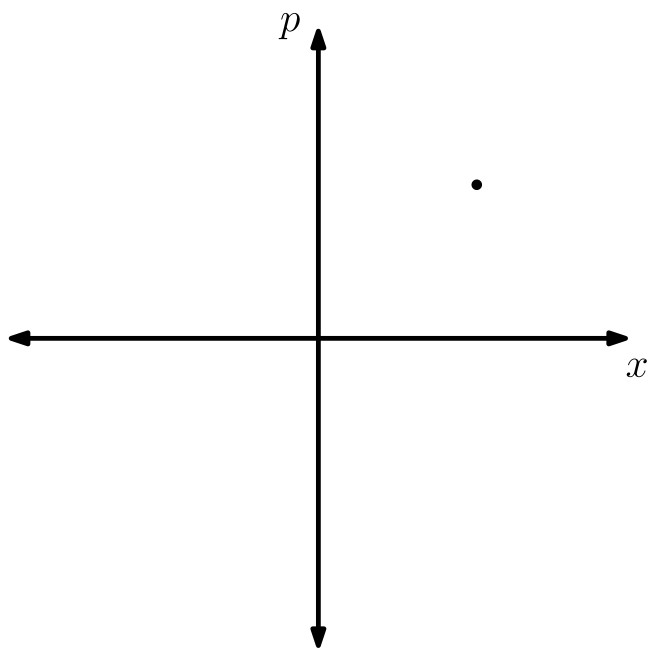
Introduction

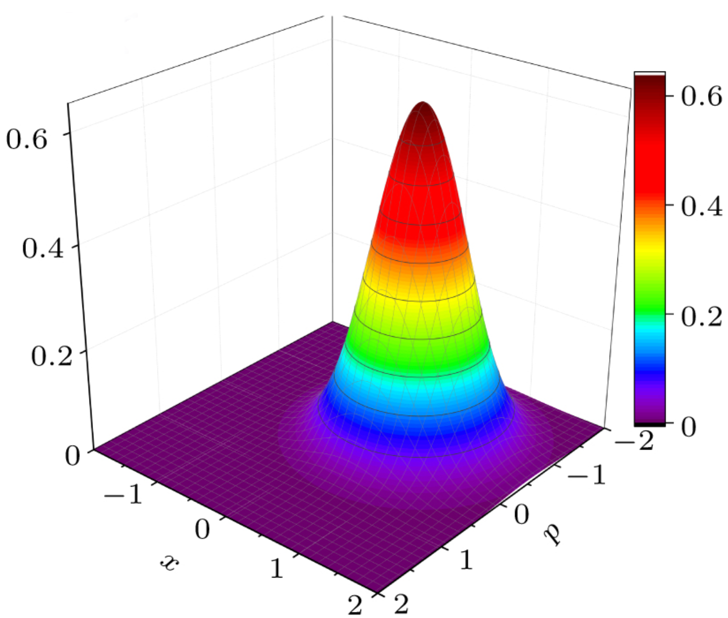
ψ-epistemic vs. ψ-ontic

- *Ontic state*: a state of reality.
  - ψ-*ontic*: the quantum state is ontic.
- *Epistemic state*: a state of knowledge or information.
  - ψ-*epistemic*: the quantum state is epistemic.
- Note: We only consider *realist* versions here, i.e. *ontological models*.

Ontic state



Epistemic state



- 2004/2007: Spekkens proposes *epistricted toy theory* in which there is a limit - *epistemic restriction* - on what we can know about the ontic state. Roughy: at most half of the available information can be known.
  - Many puzzling quantum phenomena have natural explanations in this theory because the analogues of quantum states are epistemic.
  - Extensions of the theory cover odd-dimensional stabilizer quantum mechanics and Gaussian quantum mechanics.
- 2007/2010: Harrigan and Spekkens propose formal definitions of ψ-ontic and ψ-epistemic.

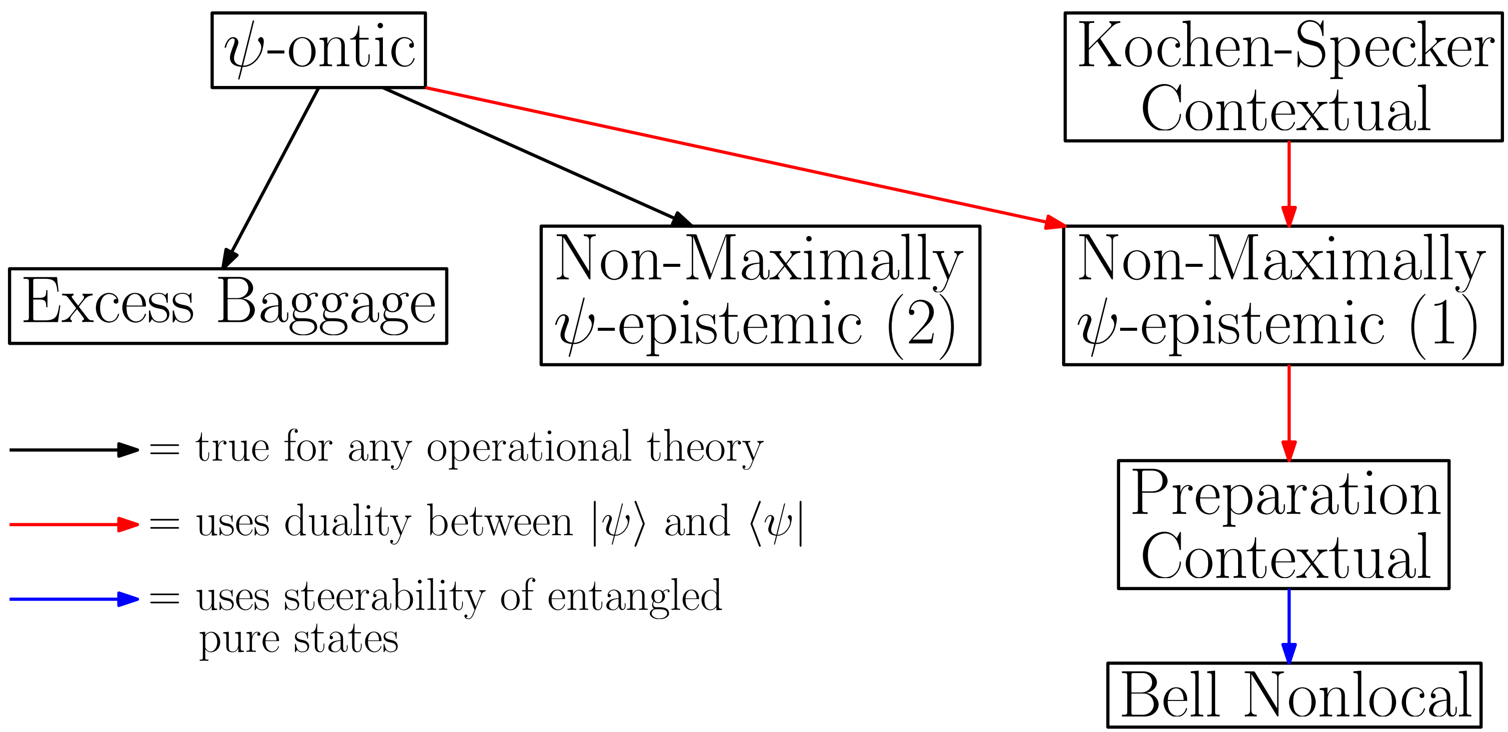
R. W. Spekkens, *Phys. Rev. A* 75(3):032110 (2007) arXiv:quant-ph/0401052
N. Harrigan & R. W. Spekkens, *Found. Phys.* 40:125 (2010) arXiv:0706.2661

Recent History of Ψ-ontology

- 2011/2012: Pusey-Barrett-Rudolph prove that an ontological model of quantum theory satisfying the *Preparation Independence Posulate (PIP)* must be ψ-ontic.
  - Other ψ-ontology theorems followed with different assumptions. The PIP alternative assumptions were criticized.
  - 2014: Some guy wrote an ovely long review article about this.
- 2012/2013: Without the PIP, ψ-epistemic models are shown to exist for all finite-dimensional Hilbert spaces.
- Two alternative tracks for ψ-ontology theorems:
  - Find alternative assumptions that are "less controversial".
  - Stick with the bare ontological models framework and prove something weaker.

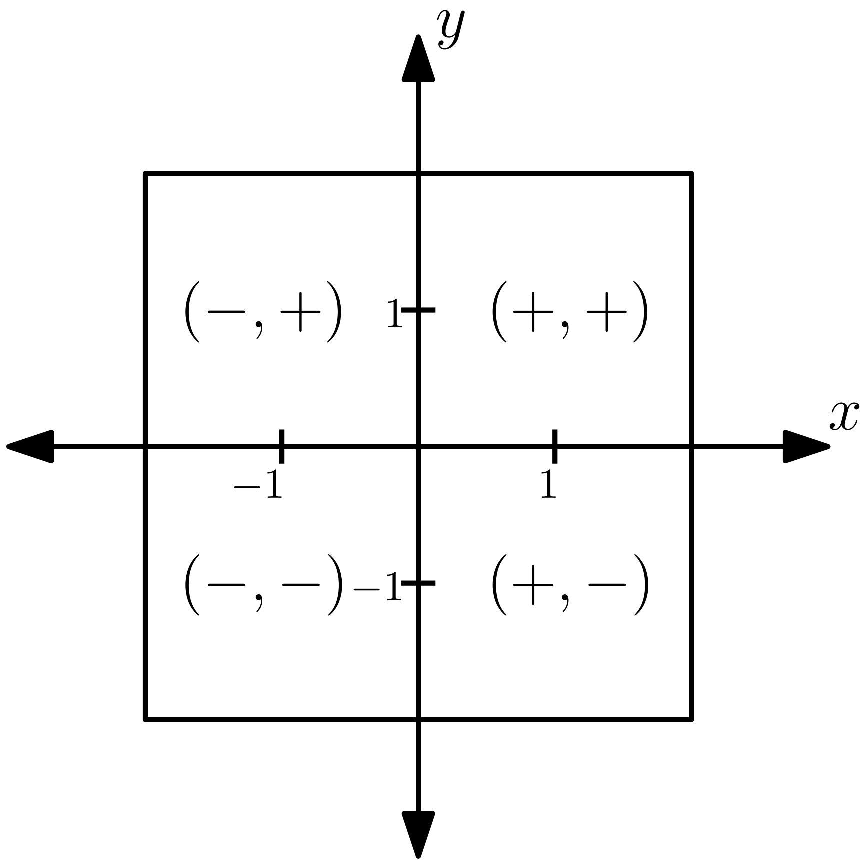
M. Pusey et. al., *Nature Physics*, 8:475–478 (2012) arXiv:1111.3328
M. Leifer, *Quanta*, 3:67–155 (2014) arXiv:1409.1570
P. G. Lewis et. al., *Phys. Rev. Lett.* 109:150404 (2012) arXiv:1201.6554, S. Aaronson et. al., *Phys. Rev. A* 88:032111 (2013) arXiv:1303.2834

Heirarchy of No-Go Results

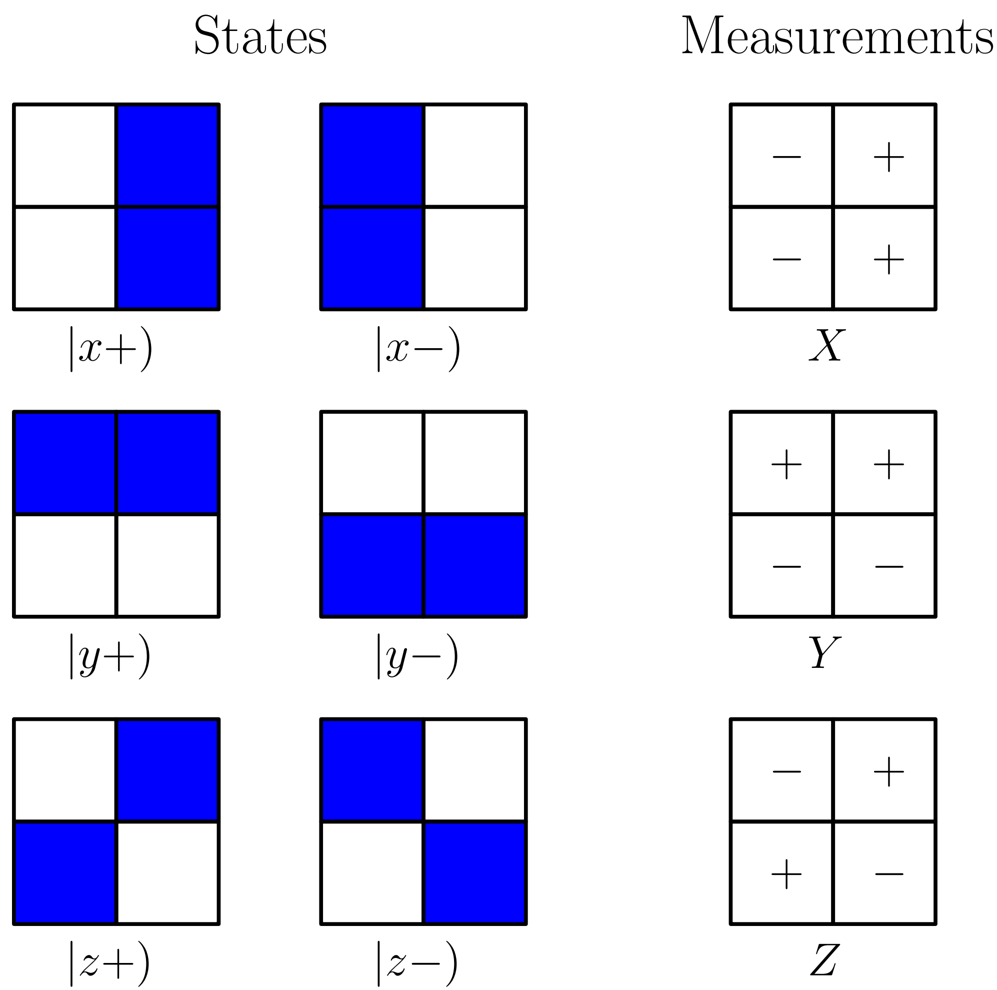


Explanations in the Spekkens Toy Theory

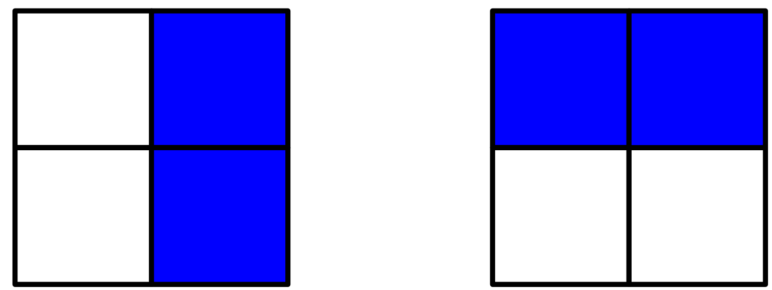
Ontology of a Spekkens toy bit



States and measurements of a toy bit



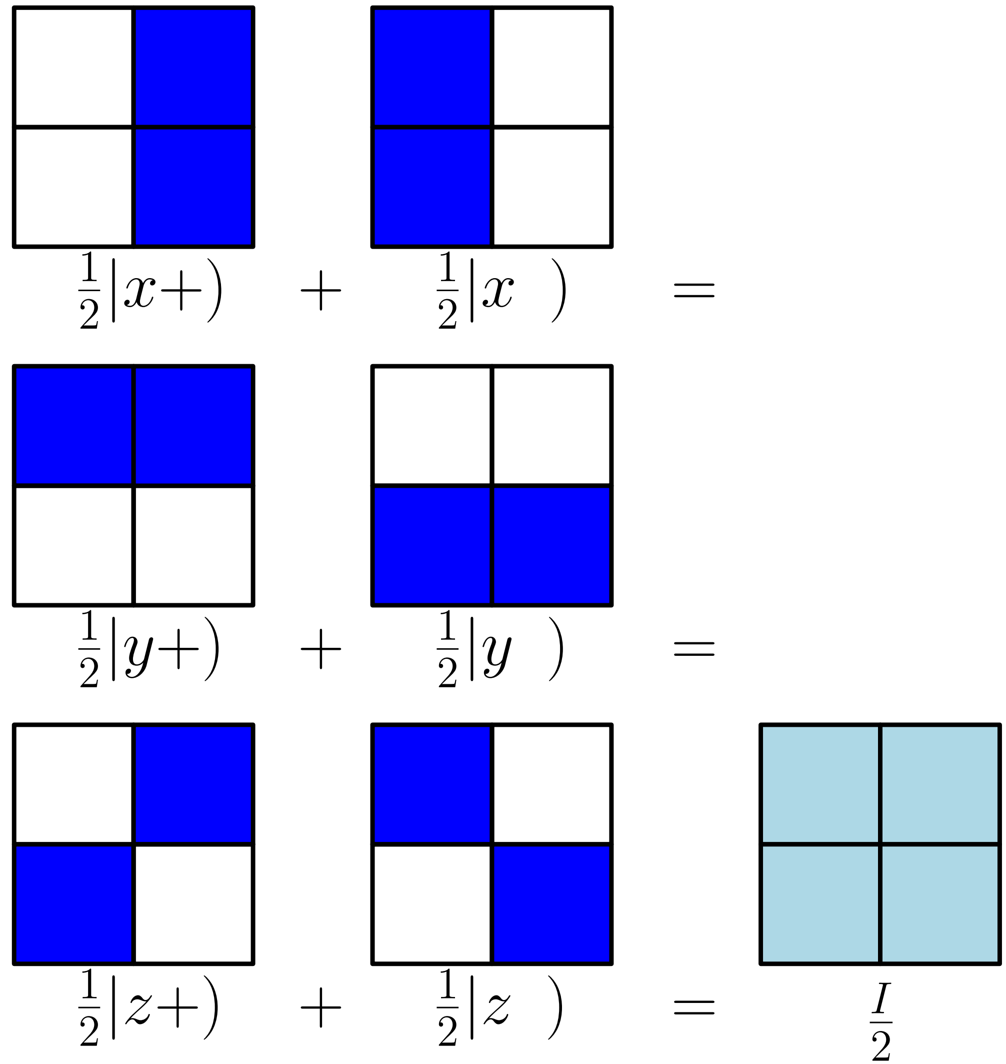
Indistinguishability of Nonorthogonal Pure States

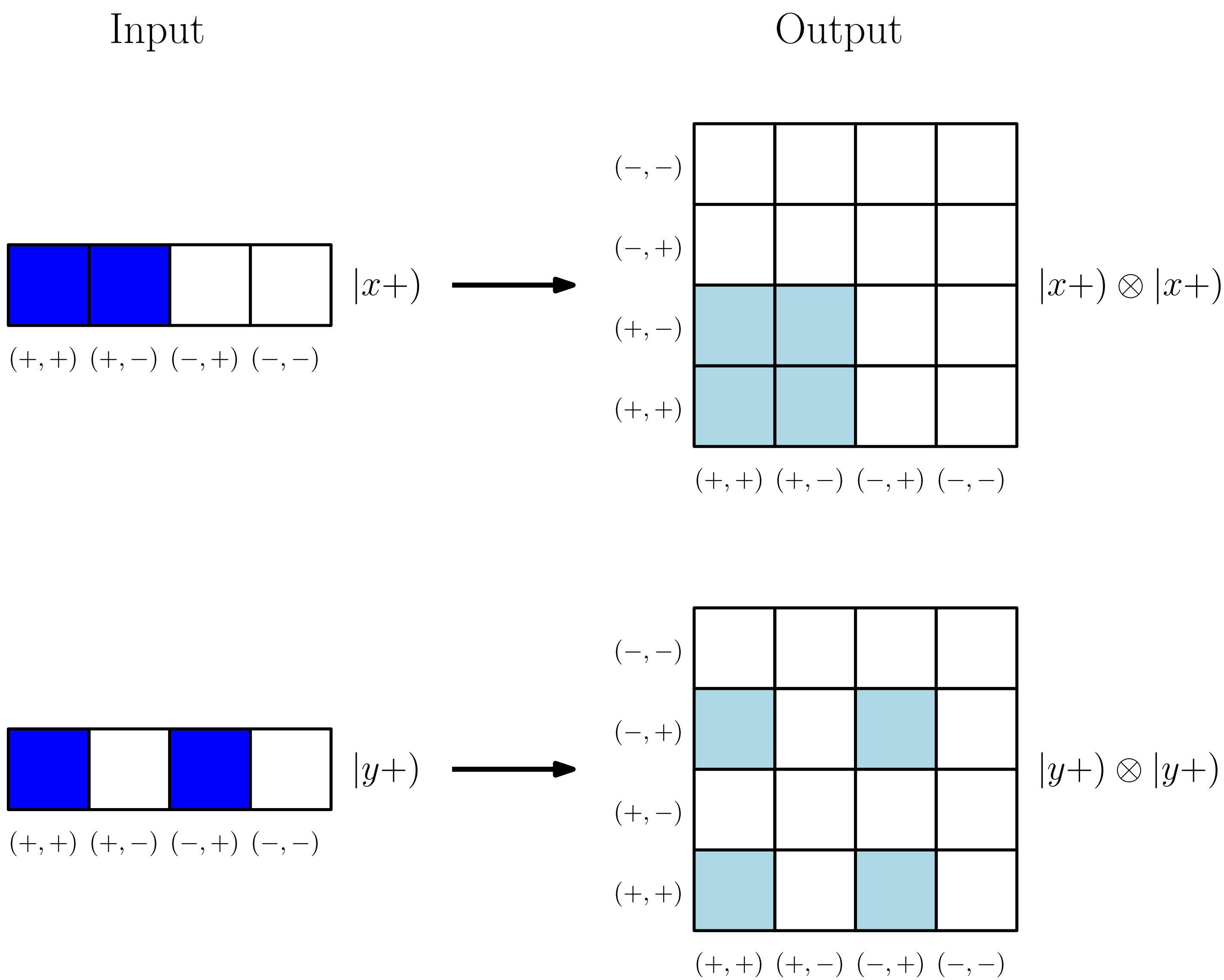


$|x+)$ $\qquad$ $|y+)$

- There can be no measurement that distinguishes $|x+)$ and $|y+)$ with certainty because, 50% of the time they both produce the same ontic state $(+,+)$.

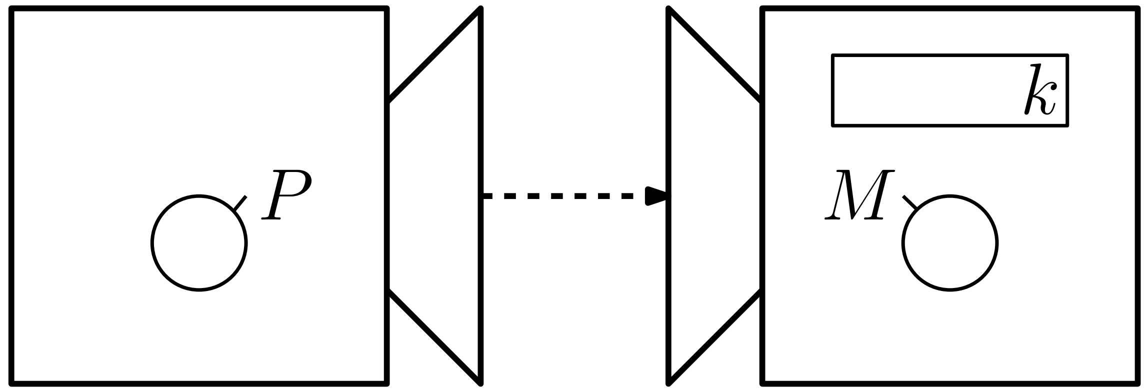
Non-uniquness of decompositions of mixed states into pure states



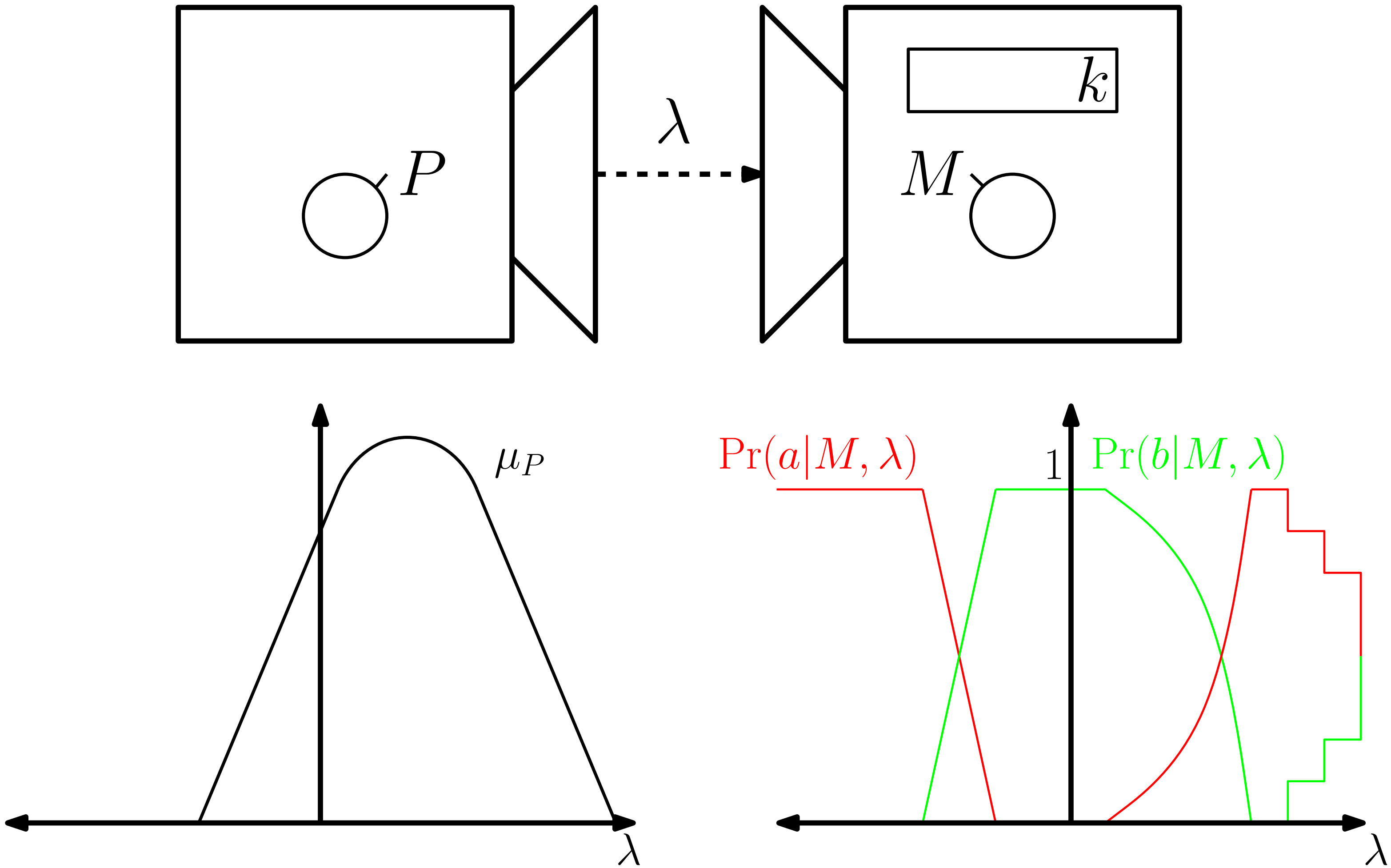


Input
Output
$|x+)$
$|x+) \otimes |x+)$
$|y+)$
$|y+) \otimes |y+)$
$(+,+)$ $(+,-)$ $(-,+)$ $(-,-)$
$(-,-)$
$(-,+)$
$(+,-)$
$(+,+)$

Ontological Models



- P is a choice of *preparation*.
- M is a choice of *measurement*.
- k is the *outcome* of the measurement.
- An operational theory assigns probabilities $\mathrm{Prob}(k|P, M)$ to each such experiment.



$$\mathrm{Prob}(k|P,M) = \int \mathrm{Pr}(k|M,\lambda)d\mu_P$$

Formal definition

An ontological model consists of:

- A measurable space (Λ, Σ).

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$$\forall\lambda, \; \sum_k \text{Pr}(k|M, \lambda) = 1.$$

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$$\forall \lambda, \ \sum_k \text{Pr}(k|M, \lambda) = 1.$$

The model is required to reproduce the operational predictions, i.e.

$$\int_\Lambda \text{Pr}(k|M, \lambda) d\mu_P = \text{Prob}(k|P, M).$$

- We are most interested in the case where the operational theory has a model within quantum theory, in which case:
- Each preparation P is assigned a density operator ρ_P.
- Each measurement M is assigned a POVM $\{E_k^M\}$, s.t.

$$\sum_k E_k^M = I.$$

- The operational probabilities are given by

$$\text{Prob}(k|P, M) = \text{Tr}\left(E_k^M \rho_P\right).$$

- and so an ontological model must satisfy

$$\text{Tr}\left(E_k^M \rho_P\right) = \int_\Lambda \text{Pr}(k|\lambda, M) \mathrm{d}\mu_P.$$

Naughty Notation

- The mappings $P \to \rho_P$ and $(M, k) \to E_k^M$ need not be one-to-one.
  - $\rho_{P_1} = \rho_{P_2}$ does not imply $\mu_{P_1} = \mu_{P_2}$.
  - $E_{k_1}^{M_1} = E_{k_2}^{M_2}$ does not imply $\text{Pr}(k_1|\lambda, M_1) = \text{Pr}(k_2|\lambda, M_2)$.
- In fact, in general, they cannot be because of contextuality.
- It is very naughty to write:
  - μ_ρ instead of μ_P,
  - $\text{Pr}(k|\lambda, E)$ instead of $\text{Pr}(k|\lambda, M)$.
- However, we will often do so to avoid clutter.
  - A statement involving μ_ρ really means:

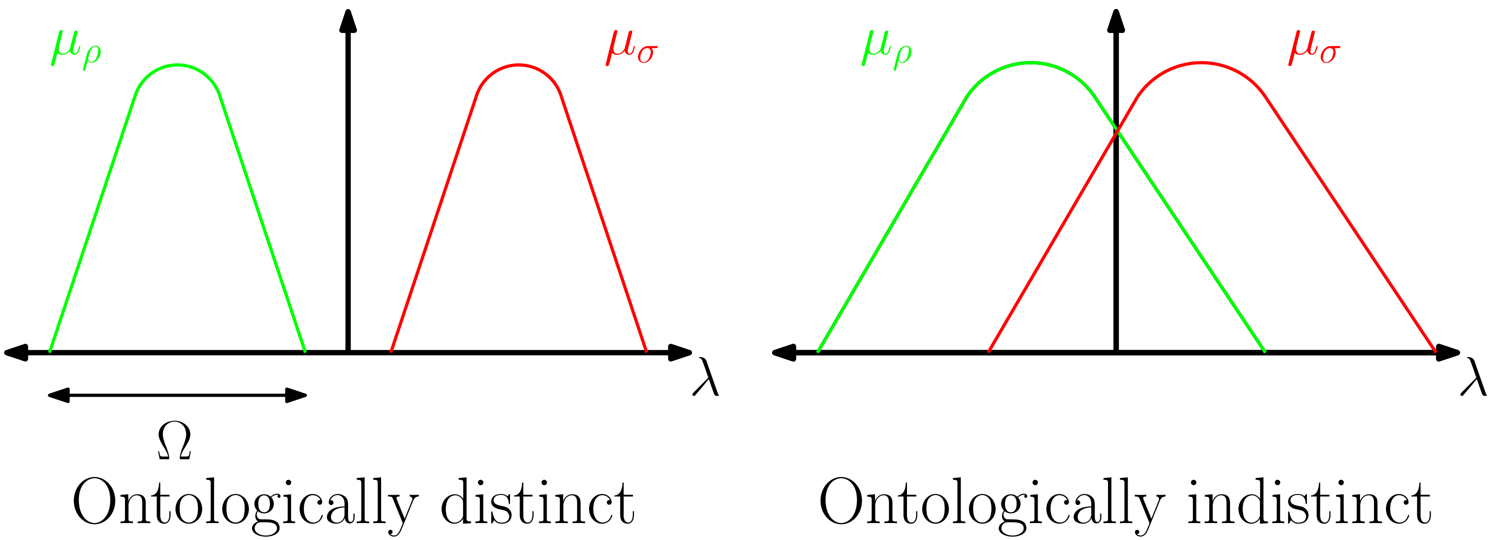
$$\forall P \quad \text{s.t.} \quad \rho_P = \rho, \ \text{ the same statement for } \mu_P.$$

  - A statement involving $\text{Pr}(k|\lambda, E)$ really means:

$$\forall (M, k) \quad \text{s.t.} \quad E_k^M = E, \ \text{ the same statement for } \text{Pr}(k|\lambda, M).$$

- ρ and σ are *ontologically distinct* in an ontological model if there exists $\Omega \in \Sigma$ s.t.

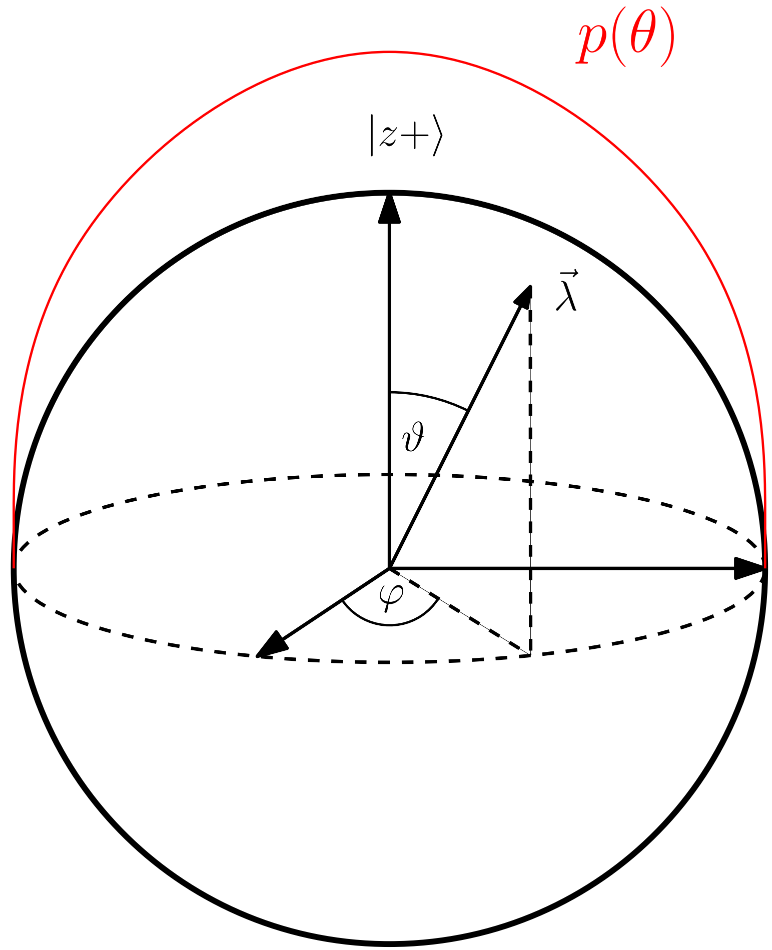
$$\mu_\rho(\Omega) = 1 \qquad \mu_\sigma(\Omega) = 0.$$



- An ontological model is *ψ-ontic* if every pair of pure states is ontologically distinct. Otherwise it is *ψ-epistemic*.

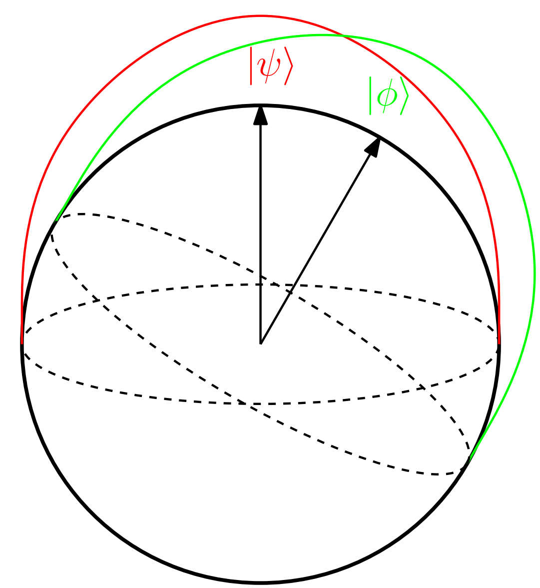
Ψ-epistemic models

The Kochen-Specker model for a qubit



$$\mu_{z+}(\Omega) = \int_\Omega p(\vartheta) \sin\vartheta d\vartheta d\varphi$$

$$p(\vartheta) = \begin{cases} \frac{1}{\pi}\cos\vartheta, & 0 \le \vartheta \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \vartheta \le \pi \end{cases}$$



S. Kochen and E. Specker, *J. Math. Mech.*, 17:59–87 (1967)

Models for arbitrary finite dimension

- Lewis et. al. provided a ψ-epistemic model for all finite d.
  - P. G. Lewis et. al., *Phys. Rev. Lett.* 109:150404 (2012) arXiv:1201.6554
- Aaronson et. al. provided a similar model in which every pair of nonorthogonal states is ontologically indistinct.
  - S. Aaronson et. al., *Phys. Rev. A* 88:032111 (2013) arXiv:1303.2834
- These models have the feature that, for a fixed inner product, the amount of overlap decreases with d.
- This invalidates the Spekkens' toy model explanations, so stronger notions of ψ-epistemic should be investigated.

Maximally ψ-epistemic models

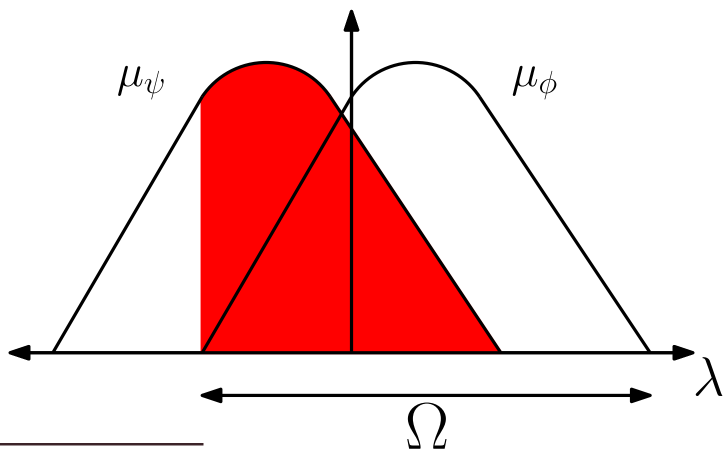
Two definitions of maximally psi-epistemic models

- Two related but nonequivalent definitions of a maximally ψ-epistemic model have been proposed.
- They are both based on the Spekkens explanation for the indistinguishability of pure states: the indistinguishability should be explained by overlap of the corresponding probability measures.

- An ontological model is *Maximally ψ-epistemic 1* if, for every pair of pure states $|\psi\rangle, |\phi\rangle$,

$$\int_{\Lambda} \text{Pr}(\phi|M,\lambda)\,\mathrm{d}\mu_{\psi} = \int_{\Omega} \text{Pr}(\phi|M,\lambda)\,\mathrm{d}\mu_{\psi},$$

for every $\Omega \in \Sigma$ such that $\mu_{\phi}(\Omega) = 1$.



O. Maroney, (2012) arXiv:1207.6906
M. Leifer and O. Maroney, *Phys. Rev. Lett.* 110:120401 (2013) arXiv:1208.5132
M. Leifer, *Quanta*, 3:67–155 (2014) arXiv:1409.1570

- The probability of obtaining $|\phi\rangle$ when the system is prepared in the state $|\phi\rangle$ is entirely explained by the overlap of μ_ψ and μ_ϕ.

- The Kochen-Specker model and Spekkens' toy theory are maximally ψ-epistemic 1.

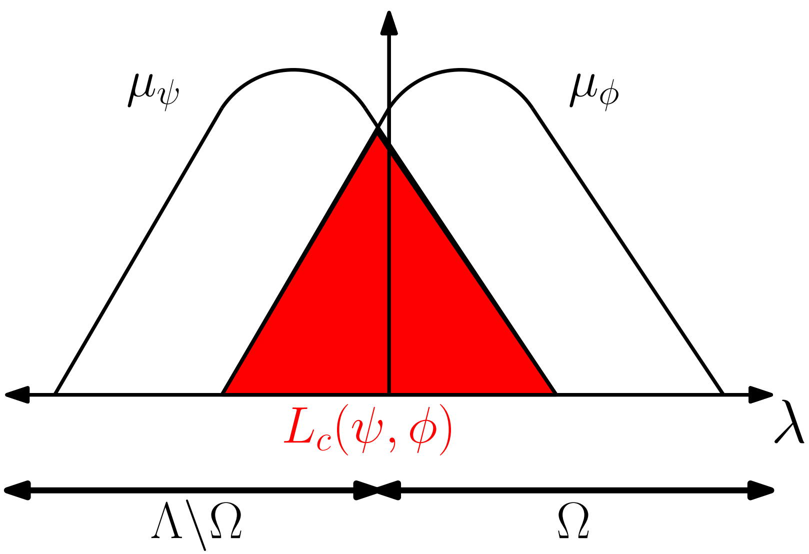
- Can show that

$$\text{Maximally } \psi\text{-epistemic 1} \quad \Rightarrow \quad \text{Kochen-Specker noncontextual,}$$

so any proof of Kochen-Specker contextuality rules out maximally ψ-epistemic 1 models.

- *Classical overlap*:

$$L_c(\psi, \phi) := \inf_{\Omega \in \Sigma} [\mu_\psi(\Omega) + \mu_\phi(\Lambda \backslash \Omega)]$$



- Optimal success probability of distinguishing $|\psi\rangle$ and $|\phi\rangle$ if you know λ:

$$p_c(\psi, \phi) = \frac{1}{2}(2 - L_c(\psi, \phi))$$

Quantum Symmetric overlap

- *Classical overlap*:

$$L_c(\psi,\phi) := \inf_{\Omega\in\Sigma}\left[\mu_\psi(\Omega) + \mu_\phi(\Lambda\backslash\Omega)\right]$$

- *Quantum overlap*:

$$\begin{aligned} L_q(\psi,\phi) &:= \inf_{0\leq E\leq I}\left[\langle\psi|\, E\, |\psi\rangle + \langle\phi|\, (I-E)\, |\phi\rangle\right] \\ &= 1-\sqrt{1-|\langle\phi|\psi\rangle|^2} \end{aligned}$$

- Optimal success probability of distinguishing $|\psi\rangle$ and $|\phi\rangle$ based on a quantum measurement:

$$p_q(\psi,\phi) = \frac{1}{2}\left(2 - L_q(\psi,\phi)\right)$$

Maximally ψ-epistemic 2

- A model is *maximally ψ-epistemic 2* if, for all pure states $|\psi\rangle$, $|\phi\rangle$,

$$L_c(\psi, \phi) = L_q(\psi, \phi).$$

- The indistinguishability of pure states is entirely explained by the fact that the corresponding probability measures overlap.
- The Kochen-Specker model and Spekkens' toy theory are maximally ψ-epistemic 2.
- Although maximally ψ-epistemic 2 does not seem to imply Kochen-Specker noncontextuality, we can use noncontextuality inequalities to rule it out, as we will see.

J. Barrrett et. al., *Phys. Rev. Lett.* 112:250403 (2014) arXiv:1310.8302

Criticism

- The structure of these definitions is as follows:
  - Take one of the explanations of quantum phenomena from the Spekkens toy theory.
  - Demand that this explanation completely accounts for the phenomenon in a maximally ψ-epistemic model.
- The decision to focus on the indistinguishability of pure states is arbitrary. We could instead demand:
  1. The optimal fidelity of approximate cloning is equal to the optimal fidelity of cloning when you know λ.
  2. Different decompositions of mixed states always give the same probability measure.
  3. Use a different state discrimination task, e.g. unambiguous discrimination.
- These give different classifications of models, e.g. 2 is preparation noncontextuality, which fails for any model of a qubit, such as the Kochen-Specker model.

Criticism

- In any ontological model there are two possible explanations of state indistinguishability:
  1. The probability measures overlap.
  2. The response functions only reveal coarse-grained information about λ.
- Why should 2 play no role in a ψ-epistemic model?
  - In Spekkens' toy theory the assumption that you can only know half of the available information about λ implies that the response functions must be coarse-grained.
  - It just so happens that Spekkens' theory is still maximally ψ-epistemic, but in general there is a principled epistemic reason for coarse-graining, and no good argument for why this should play no role in state indistinguishability.

- In light of this, the goal ought to be: Show that there must be pure states $|\psi\rangle$, $|\phi\rangle$ such that

$$L_c(\psi, \phi) \leq \epsilon, \qquad L_q(\psi, \phi) > 1 - \delta,$$

with $\epsilon, \delta > 0$ and try to make ϵ and δ as small as possible.

Overlap bounds

- Given a set V of states, and another state $|\psi\rangle$, we can upper bound the average overlap

$$\langle L_c \rangle = \sum_{|a\rangle \in V} p_a L_c(\psi, a),$$

where p_a is a probability distribution over V.

- Most works use this to bound the ratio:

$$k = \frac{\langle L_c \rangle}{\langle L_q \rangle}.$$

- Better to use the difference:
  - *Overlap deficit*: $\Delta L = \langle L_q \rangle - \langle L_c \rangle$

	Dimension	$\|V\|$	$\langle L_c \rangle$	$\langle L_q \rangle$
Barrett et. al.[1]	Prime power $d \geq 4$	d^2	$1/d^2$	$1 - \sqrt{1 - 1/d}$
Leifer[2]	$d \geq 3$	2^{d-1}	$1/2^{d-1}$	$1 - \sqrt{1 - 1/d}$
Branciard[3]	$d \geq 4$	$n \geq 2$	$1/n$	$1 - \sqrt{1 - \frac{1}{4} n^{-1/(d-2)}}$
Amaral et. al.[4]	$d \geq n_j$	$n_j \geq ?$	$n_j^{\delta - 1}$	$1 - \sqrt{\frac{1}{2} + \epsilon}$

[1] J. Barrrett et. al., Phys. Rev. Lett. 112, 250403 (2014)
[2] ML, Phys. Rev. Lett. 112, 160404 (2014)
[3] C. Branciard, Phys. Rev. Lett. 113, 020409 (2014)
[4] B. Amaral et. al., Phys. Rev. A 92, 062125 (2015)

Optimizing for distinguishability deficit

	Optimal dimension	Optimal $\vert V\vert$	ΔL
Barrett et. al.	4	16	0.0715
Leifer	7	64	0.0586
Branciard	4	$n \to \infty$	0.134
Amaral et. al.	$d \to \infty$	$n_j \to \infty$	0.293

- Ringbauer et. al.[5] experiment (based on Branciard's construction) obtained:

$$k \leq 0.690 \pm 0.001$$

$$\Delta L \geq 0.047 \pm 0.010$$

- My analysis suggests larger ΔL should be obtainable from the Barrett et. al. construction.

[5] M. Ringbauer et. al. Nature Physics 11, 249–254 (2015).

Overlap bounds from contextuality

Kochen-Specker noncontextuality

- Let $\mathcal{M}$ be a set of orthonormal bases in $\mathbb{C}^d$.
- An ontological model for $\mathcal{M}$ is *Kochen Specker noncontextual* if it is
  - *Outcome deterministic*: $\text{Pr}(a|M,\lambda) \in \{0,1\}$
  - *Measurement noncontextual*: If there exist $M, N \in \mathcal{M}$ and $|a\rangle$ such that $|a\rangle \in M$ and $|a\rangle \in N$ then

$$\text{Pr}(a|M,\cdot) = \text{Pr}(a|N,\cdot).$$

- Define:

$$\Gamma_a^M = \{\lambda \in \Lambda | \text{Pr}(a|M,\lambda) = 1\} \qquad \Gamma_a = \bigcap_{\{M \in \mathcal{M} | |a\rangle \in M\}} \Gamma_a^M$$

Theorem: There exists a KS noncontextual model for $\mathcal{M}$ iff there exists a model where, for all $|\psi\rangle$, $M \in \mathcal{M}$, $|a\rangle \in M$,

$$\int_\Lambda \text{Pr}(a|M,\lambda) d\mu_\psi(\lambda) = \mu_\psi(\Gamma_a).$$

- For a (finite) set V of states, a noncontextuality inequality is a bound of the form

$$\sum_{|a\rangle \in V} p_a \mu_\psi(\Gamma_a) \leq \gamma.$$

- Let $\mathcal{M}$ be a covering set of bases for V. We have

$$\int_\Lambda \text{Pr}(a|M,\lambda) d\mu_a(\lambda) = |\langle a|a\rangle|^2 = 1$$

and since $\text{Pr}(a|M,\lambda) \leq 1$ this implies that $\mu_a(\Gamma_a^M) = 1$.

- Since $\Gamma_a = \cap_{M \in \mathcal{M} | |a\rangle \in M} \Gamma_a^M$ is a finite intersection of measure one sets, we also have

$$\mu_a(\Gamma_a) = 1.$$

Deriving overlap bounds

- Now,

$$L_c(\psi, a) = \inf_{\Omega \in \Sigma} [\mu_\psi(\Omega) + \mu_a(\Lambda \backslash \Omega)]$$
$$\leq \mu_\psi(\Gamma_a) + \mu_a(\Lambda \backslash \Gamma_a)$$

- We just showed that $\mu_a(\Gamma_a) = 1$, so $\mu_a(\Lambda \backslash \Gamma_a) = 0$, and hence

$$L_c(\psi, a) \leq \mu_\psi(\Gamma_a).$$

- Hence,

$$\sum_{|a\rangle \in V} p_a L_c(\psi, a) \leq \sum_{|a\rangle \in V} p_a \mu_\psi(\Gamma_a) \leq \gamma.$$

- Using Cabello, Severini and Winter's results[6], for a set of states V, we can derive

$$\frac{1}{|V|}\sum_{|a\rangle\in V} L_c(\psi, a) \leq \frac{\alpha(G)}{|V|},$$

where $\alpha(G)$ is the *independence number* of the *orthogonality graph* of V.

- Other bounds come from a different technique, introduced by Barrett et. al.[7].

- It turns out that this method is also based on noncontextuality inequalities[8].

[6] A. Cabello, S. Severini, A. Winter, Phys. Rev. Lett. 112:040401 (2014).
[7] J. Barrrett et. al., Phys. Rev. Lett. 112, 250403 (2014)
[8] M. Leifer & C. Duarte, *Phys. Rev. A* 101:062113 (2020) arXiv:2001.11485

Conclusions

Related Work

- Bounds involving dynamics:
  - Based on unitary dynamics: J. Allen,Quantum Stud.: Math. Found. 3, 161–177 (2016).
  - Based on collapse dynamics: J. Ruebeck et. al., Quantum 4, 242 (2020).
- Bounds based on higher order overlaps:
  - S. Ray, R. Visweshwaran, D. Saha, arXiv:2401.17980 (2024).
- Several posters here.

Summary and Open questions

- Summary:
  - Several bounds exist showing $k \to 0$. Harder to get $\Delta L \approx 1$. Best current bound is $\Delta L \approx 0.293$.
  - Any noncontextuality inequality of the appropriate type is an overlap bound.
- Open questions:
  - Error analysis for arbitrary noncontextuality-based overlap bounds.
  - What is the best possible bound on ΔL?
  - Are their overlap bounds that do not follow from noncontextuality inequalities?
  - Applications in quantum information.

What now for ψ-epistemicists?

In any case, ontological models are not viable for a ψ-epistemicist, so what now?

- Become neo-Copenhagen.
- Adopt a more exotic ontology:
  - Nonstandard logics and probability theories.
  - Ironic many-worlds.
  - Retrocausality.
  - Relationalism.

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  - Ironic many-worlds.
  - Retrocausality.
  - Relationalism.
- Explanatory conservatism: If there is a natural explanation for a quantum phenomenon then we should adopt an interpretation that incorporates it.
  - Suggests exploring exotic ontologies.

References

- Review articles:
  - ML, "Is the quantum state real? An extended review of ψ-ontology theorems", *Quanta* 3:67–155 (2014), arXiv:1409.1570.
  - D. Jennings and ML, "No Return to Classical Reality", *Contemp. Phys.* 56 (2015). arXiv:1501.03202.
- Overlap bounds and contextuality:
  - ML and O. Maroney, "Maximally epistemic interpretations of the quantum state and contextuality", *Phys. Rev. Lett.* 110:120401 (2013) arXiv:1208.5132.
  - ML, "ψ-epistemic models are exponentially bad at explaining the distinguishability of quantum states "*Phys. Rev. Lett.* 112:160404 (2014) arXiv:1401.7996.
  - ML, "Review of Overlap Bounds on the Reality of the Quantum State" in preparation.

Additional slides

Other arguments

- Collapse of the wavefunction
- Generalized probability theory
- Excess baggage

See ML Quanta 3:67–155 (2014) for more details.

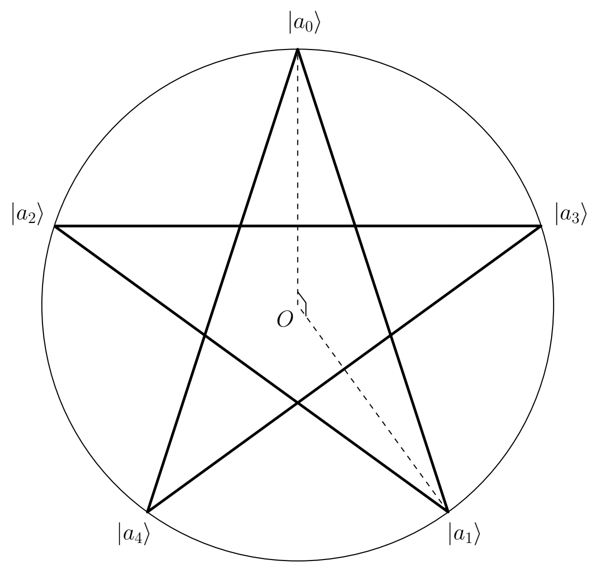
Arguments for ontic quantum states

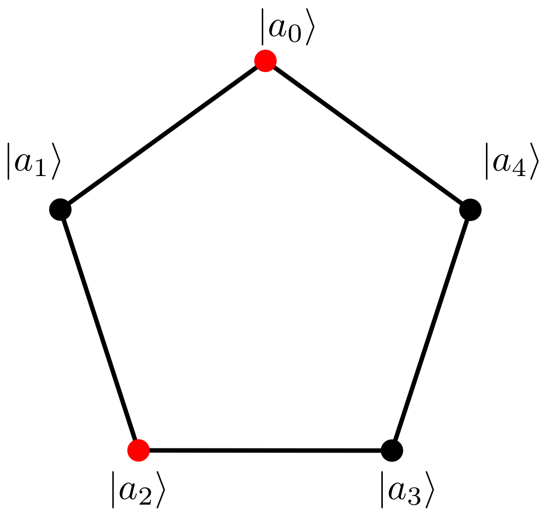
- Interference
- Eigenvalue-eigenstate link
- Lack of imagination
- Quantum computing

See ML Quanta 3:67–155 (2014) for more details.

Example: Klyachko inequality

- Ususal proofs of contextuality inequalities use $\Gamma_a \cap \Gamma_b = \emptyset$ when $|\langle a|b\rangle|^2 = 0$.
- Example:
  - $|a_j\rangle = \sin\vartheta\cos\varphi_j\,|0\rangle + \sin\vartheta\sin\varphi_j\,|1\rangle + \cos\vartheta\,|2\rangle$
  - $\varphi_j = \frac{4\pi j}{5}$ and $\cos\vartheta = \frac{1}{\sqrt[4]{5}}$





$$\langle L_c\rangle = \frac{1}{5}\sum_{j=0}^{4} L_c(a_j,\psi) \leq \frac{2}{5}$$