Singular Levi-flat hypersurfaces (2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Small review:

M given by $\{r = 0\}$.

The full Hessian is

$$H_p = \begin{bmatrix} \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_n}\Big|_p & \frac{\partial^2 r}{\partial \bar{z}_1 \partial \bar{z}_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_1 \partial \bar{z}_n}\Big|_p \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 r}{\partial \bar{z}_n \partial z_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_n}\Big|_p & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_n}\Big|_p \\ \frac{\partial^2 r}{\partial z_1 \partial z_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial z_1 \partial z_n}\Big|_p & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_n}\Big|_p \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 r}{\partial z_n \partial z_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n}\Big|_p & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_1}\Big|_p & \cdots & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_n}\Big|_p \end{bmatrix} = \begin{bmatrix} L_p & \overline{Z_p} \\ Z_p & L_p^t \end{bmatrix}$$

L_p is the complex Hessian.

$X_p^* L_p X_p$ for $X_p \in T_p^{(1,0)} M$ is the Levi form.

We are mostly interested in biholomorphic invariants.
So what happens under a biholomorphism?

We are mostly interested in biholomorphic invariants.
So what happens under a biholomorphism?

Suppose $f\colon V \to V'$ is biholomorphic,
$r\colon V' \to \mathbb{R}$ a defining function for $M \subset V'$
and $r \circ f$ the defining function for $f^{-1}(M) \subset V$.

We are mostly interested in biholomorphic invariants.
So what happens under a biholomorphism?

Suppose $f\colon V \to V'$ is biholomorphic,
$r\colon V' \to \mathbb{R}$ a defining function for $M \subset V'$
and $r \circ f$ the defining function for $f^{-1}(M) \subset V$.
Let z denote coordinates in V and ζ in V'

We are mostly interested in biholomorphic invariants.
So what happens under a biholomorphism?

Suppose $f\colon V \to V'$ is biholomorphic,
$r\colon V' \to \mathbb{R}$ a defining function for $M \subset V'$
and $r \circ f$ the defining function for $f^{-1}(M) \subset V$.
Let z denote coordinates in V and ζ in V'

First the Z matrix:

$$\frac{\partial^2 (r \circ f)}{\partial z_k \partial z_\ell}$$

We are mostly interested in biholomorphic invariants.
So what happens under a biholomorphism?

Suppose $f\colon V \to V'$ is biholomorphic,
$r\colon V' \to \mathbb{R}$ a defining function for $M \subset V'$
and $r \circ f$ the defining function for $f^{-1}(M) \subset V$.
Let z denote coordinates in V and ζ in V'

First the Z matrix:

$$\frac{\partial^2 (r \circ f)}{\partial z_k \partial z_\ell} = \frac{\partial}{\partial z_k} \sum_{m=1}^{n} \left(\frac{\partial r}{\partial \zeta_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \frac{\partial f_m}{\partial z_\ell}\bigg|_{z} + \frac{\partial r}{\partial \bar{\zeta}_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \overbrace{\frac{\partial \bar{f}_m}{\partial z_\ell}\bigg|_{\bar{z}}}^{0} \right)$$

We are mostly interested in biholomorphic invariants.
So what happens under a biholomorphism?

Suppose $f\colon V \to V'$ is biholomorphic,
$r\colon V' \to \mathbb{R}$ a defining function for $M \subset V'$
and $r \circ f$ the defining function for $f^{-1}(M) \subset V$.
Let z denote coordinates in V and ζ in V'

First the Z matrix:

$$\frac{\partial^2 (r \circ f)}{\partial z_k \partial z_\ell} = \frac{\partial}{\partial z_k} \sum_{m=1}^{n} \left(\frac{\partial r}{\partial \zeta_m}\Big|_{(f(z),\bar{f}(\bar{z}))} \frac{\partial f_m}{\partial z_\ell}\Big|_{z} + \frac{\partial r}{\partial \bar{\zeta}_m}\Big|_{(f(z),\bar{f}(\bar{z}))} \underbrace{\frac{\partial \bar{f}_m}{\partial z_\ell}\Big|_{\bar{z}}}_{0} \right)$$

$$= \sum_{m,\nu=1}^{n} \left(\frac{\partial^2 r}{\partial \zeta_\nu \partial \zeta_m}\Big|_{(f(z),\bar{f}(\bar{z}))} \frac{\partial f_\nu}{\partial z_k}\Big|_{z} \frac{\partial f_m}{\partial z_\ell}\Big|_{z} + \frac{\partial^2 r}{\partial \bar{\zeta}_\nu \partial \zeta_m}\Big|_{(f(z),\bar{f}(\bar{z}))} \underbrace{\frac{\partial \bar{f}_\nu}{\partial z_k}\Big|_{\bar{z}}}_{0} \frac{\partial f_m}{\partial z_\ell}\Big|_{z} \right)$$

$$+ \sum_{m=1}^{n} \frac{\partial r}{\partial \zeta_m}\Big|_{(f(z),\bar{f}(\bar{z}))} \frac{\partial^2 f_m}{\partial z_k \partial z_\ell}\Big|_{z}$$

We are mostly interested in biholomorphic invariants.
So what happens under a biholomorphism?

Suppose $f \colon V \to V'$ is biholomorphic,
$r \colon V' \to \mathbb{R}$ a defining function for $M \subset V'$
and $r \circ f$ the defining function for $f^{-1}(M) \subset V$.
Let z denote coordinates in V and ζ in V'

First the Z matrix:

$$\frac{\partial^2 (r \circ f)}{\partial z_k \partial z_\ell} = \frac{\partial}{\partial z_k} \sum_{m=1}^{n} \left(\frac{\partial r}{\partial \zeta_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \frac{\partial f_m}{\partial z_\ell}\bigg|_{z} + \frac{\partial r}{\partial \bar{\zeta}_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \cancelto{0}{\frac{\partial \bar{f}_m}{\partial z_\ell}}\bigg|_{\bar{z}} \right)$$

$$= \sum_{m,\nu=1}^{n} \left(\frac{\partial^2 r}{\partial \zeta_\nu \partial \zeta_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \frac{\partial f_\nu}{\partial z_k}\bigg|_{z} \frac{\partial f_m}{\partial z_\ell}\bigg|_{z} + \frac{\partial^2 r}{\partial \bar{\zeta}_\nu \partial \zeta_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \cancelto{0}{\frac{\partial \bar{f}_\nu}{\partial z_k}}\bigg|_{\bar{z}} \frac{\partial f_m}{\partial z_\ell}\bigg|_{z} \right)$$

$$+ \sum_{m=1}^{n} \frac{\partial r}{\partial \zeta_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \frac{\partial^2 f_m}{\partial z_k \partial z_\ell}\bigg|_{z}$$

$$= \sum_{m,\nu=1}^{n} \frac{\partial^2 r}{\partial \zeta_\nu \partial \zeta_m} \frac{\partial f_\nu}{\partial z_k} \frac{\partial f_m}{\partial z_\ell} + \sum_{m=1}^{n} \frac{\partial r}{\partial \zeta_m} \frac{\partial^2 f_m}{\partial z_k \partial z_\ell}$$

We are mostly interested in biholomorphic invariants.
So what happens under a biholomorphism?

Suppose $f\colon V \to V'$ is biholomorphic,
$r\colon V' \to \mathbb{R}$ a defining function for $M \subset V'$
and $r \circ f$ the defining function for $f^{-1}(M) \subset V$.
Let z denote coordinates in V and ζ in V'

First the Z matrix:

$$\frac{\partial^2 (r\circ f)}{\partial z_k \partial z_\ell} = \frac{\partial}{\partial z_k} \sum_{m=1}^{n} \left(\frac{\partial r}{\partial \zeta_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \frac{\partial f_m}{\partial z_\ell}\bigg|_z + \frac{\partial r}{\partial \bar{\zeta}_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \underbrace{\frac{\partial \bar{f}_m}{\partial z_\ell}}_{0}\bigg|_{\bar{z}} \right)$$

$$= \sum_{m,\nu=1}^{n} \left(\frac{\partial^2 r}{\partial \zeta_\nu \partial \zeta_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \frac{\partial f_\nu}{\partial z_k}\bigg|_z \frac{\partial f_m}{\partial z_\ell}\bigg|_z + \frac{\partial^2 r}{\partial \bar{\zeta}_\nu \partial \zeta_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \underbrace{\frac{\partial \bar{f}_\nu}{\partial z_k}}_{0}\bigg|_{\bar{z}} \frac{\partial f_m}{\partial z_\ell}\bigg|_z \right)$$

$$+ \sum_{m=1}^{n} \frac{\partial r}{\partial \zeta_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \frac{\partial^2 f_m}{\partial z_k \partial z_\ell}\bigg|_z$$

$$= \sum_{m,\nu=1}^{n} \frac{\partial^2 r}{\partial \zeta_\nu \partial \zeta_m} \frac{\partial f_\nu}{\partial z_k} \frac{\partial f_m}{\partial z_\ell} + \boxed{\sum_{m=1}^{n} \frac{\partial r}{\partial \zeta_m} \frac{\partial^2 f_m}{\partial z_k \partial z_\ell}} \leftarrow \text{does not transform nicely}$$

Now the L_p matrix:

$$\frac{\partial^2 (r \circ f)}{\partial \bar{z}_k \partial z_\ell}$$

Now the L_p matrix:

$$\frac{\partial^2 (r \circ f)}{\partial \bar{z}_k \partial z_\ell} = \frac{\partial}{\partial \bar{z}_k} \sum_{m=1}^{n} \left(\frac{\partial r}{\partial \zeta_m}\bigg|_{(f(z),\bar{f}(\bar{z}))} \frac{\partial f_m}{\partial z_\ell}\bigg|_z \right)$$

Now the L_p matrix:

$$\frac{\partial^2 (r \circ f)}{\partial \bar{z}_k \partial z_\ell} = \frac{\partial}{\partial \bar{z}_k} \sum_{m=1}^{n} \left(\left. \frac{\partial r}{\partial \zeta_m} \right|_{(f(z), \bar{f}(\bar{z}))} \left. \frac{\partial f_m}{\partial z_\ell} \right|_z \right)$$

$$= \sum_{m,\nu=1}^{n} \left. \frac{\partial^2 r}{\partial \bar{\zeta}_\nu \partial \zeta_m} \right|_{(f(z), \bar{f}(\bar{z}))} \left. \frac{\partial \bar{f}_\nu}{\partial \bar{z}_k} \right|_{\bar{z}} \left. \frac{\partial f_m}{\partial z_\ell} \right|_z + \sum_{m=1}^{n} \left. \frac{\partial r}{\partial \zeta_m} \right|_{(f(z), \bar{f}(\bar{z}))} \cancelto{0}{\left. \frac{\partial^2 f_m}{\partial \bar{z}_k \partial z_\ell} \right|_z}$$

Now the L_p matrix:

$$\frac{\partial^2(r \circ f)}{\partial \bar{z}_k \partial z_\ell} = \frac{\partial}{\partial \bar{z}_k} \sum_{m=1}^{n} \left(\left.\frac{\partial r}{\partial \zeta_m}\right|_{(f(z),\bar{f}(\bar{z}))} \left.\frac{\partial f_m}{\partial z_\ell}\right|_z \right)$$

$$= \sum_{m,\nu=1}^{n} \left.\frac{\partial^2 r}{\partial \bar{\zeta}_\nu \partial \zeta_m}\right|_{(f(z),\bar{f}(\bar{z}))} \left.\frac{\partial \bar{f}_\nu}{\partial \bar{z}_k}\right|_{\bar{z}} \left.\frac{\partial f_m}{\partial z_\ell}\right|_z + \sum_{m=1}^{n} \left.\frac{\partial r}{\partial \zeta_m}\right|_{(f(z),\bar{f}(\bar{z}))} \underbrace{\left.\frac{\partial^2 f_m}{\partial \bar{z}_k \partial z_\ell}\right|_z}_{\to 0}$$

$$= \sum_{m,\nu=1}^{n} \frac{\partial^2 r}{\partial \bar{\zeta}_\nu \partial \zeta_m} \frac{\partial \bar{f}_\nu}{\partial \bar{z}_k} \frac{\partial f_m}{\partial z_\ell}$$

Now the L_p matrix:

$$\frac{\partial^2(r\circ f)}{\partial\bar z_k\partial z_\ell} = \frac{\partial}{\partial\bar z_k}\sum_{m=1}^{n}\left(\left.\frac{\partial r}{\partial\zeta_m}\right|_{(f(z),\bar f(\bar z))}\left.\frac{\partial f_m}{\partial z_\ell}\right|_z\right)$$

$$= \sum_{m,\nu=1}^{n}\left.\frac{\partial^2 r}{\partial\bar\zeta_\nu\partial\zeta_m}\right|_{(f(z),\bar f(\bar z))}\left.\frac{\partial\bar f_\nu}{\partial\bar z_k}\right|_{\bar z}\left.\frac{\partial f_m}{\partial z_\ell}\right|_z + \sum_{m=1}^{n}\left.\frac{\partial r}{\partial\zeta_m}\right|_{(f(z),\bar f(\bar z))}\left.\underbrace{\frac{\partial^2 f_m}{\partial\bar z_k\partial z_\ell}}_{0}\right|_z$$

$$= \sum_{m,\nu=1}^{n}\frac{\partial^2 r}{\partial\bar\zeta_\nu\partial\zeta_m}\frac{\partial\bar f_\nu}{\partial\bar z_k}\frac{\partial f_m}{\partial z_\ell}$$

$\Rightarrow$ L_p changes by $*$-congruence:

Now the L_p matrix:

$$\frac{\partial^2 (r \circ f)}{\partial \bar{z}_k \partial z_\ell} = \frac{\partial}{\partial \bar{z}_k} \sum_{m=1}^{n} \left(\left.\frac{\partial r}{\partial \zeta_m}\right|_{(f(z),\bar{f}(\bar{z}))} \left.\frac{\partial f_m}{\partial z_\ell}\right|_z \right)$$

$$= \sum_{m,\nu=1}^{n} \left.\frac{\partial^2 r}{\partial \bar{\zeta}_\nu \partial \zeta_m}\right|_{(f(z),\bar{f}(\bar{z}))} \left.\frac{\partial \bar{f}_\nu}{\partial \bar{z}_k}\right|_{\bar{z}} \left.\frac{\partial f_m}{\partial z_\ell}\right|_z + \sum_{m=1}^{n} \left.\frac{\partial r}{\partial \zeta_m}\right|_{(f(z),\bar{f}(\bar{z}))} \underbrace{\left.\frac{\partial^2 f_m}{\partial \bar{z}_k \partial z_\ell}\right|_z}_{\to 0}$$

$$= \sum_{m,\nu=1}^{n} \frac{\partial^2 r}{\partial \bar{\zeta}_\nu \partial \zeta_m} \frac{\partial \bar{f}_\nu}{\partial \bar{z}_k} \frac{\partial f_m}{\partial z_\ell}$$

$\Rightarrow$ L_p changes by $*$-congruence:

$\Rightarrow$ inertia of the Levi-form is a biholomorphic invariant!

If for all $X_p \in T_p^{(1,0)}M$, $X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k}\Big|_p$,

$$X_p^* L_p X_p = \sum_{k=1,\ell=1}^{n} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell}\Big|_p \geq 0,$$

then M is said to be *pseudoconvex* at p.

If for all $X_p \in T_p^{(1,0)}M$, $X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k}\Big|_p$,

$$X_p^* L_p X_p = \sum_{k=1,\ell=1}^{n} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell}\Bigg|_p \geq 0,$$

then M is said to be *pseudoconvex* at p.

strictly or *strongly pseucodonvex* if $X_p^* L_p X_p > 0$ for $X_p \neq 0$.

If for all $X_p \in T_p^{(1,0)}M$, $X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k}\Big|_p$,

$$X_p^* L_p X_p = \sum_{k=1,\ell=1}^{n} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell}\Big|_p \geq 0,$$

then M is said to be *pseudoconvex* at p.

strictly or *strongly pseucodonvex* if $X_p^* L_p X_p > 0$ for $X_p \neq 0$.

Note the similarity of the definition to classical convexity.

If for all $X_p \in T_p^{(1,0)}M$, $X_p = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k}\Big|_p$,

$$X_p^* L_p X_p = \sum_{k=1,\ell=1}^{n} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell}\Big|_p \geq 0,$$

then M is said to be *pseudoconvex* at p.

strictly or *strongly pseucodonvex* if $X_p^* L_p X_p > 0$ for $X_p \neq 0$.

Note the similarity of the definition to classical convexity.

Really, it is one side of the hypersurface that is pseudoconvex.

If for all $X_p \in T_p^{(1,0)}M$, $X_p = \sum_{k=1}^n a_k \frac{\partial}{\partial z_k}\big|_p$,

$$X_p^* L_p X_p = \sum_{k=1,\ell=1}^{n} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell}\bigg|_p \geq 0,$$

then M is said to be *pseudoconvex* at p.

strictly or *strongly pseucodonvex* if $X_p^* L_p X_p > 0$ for $X_p \neq 0$.

Note the similarity of the definition to classical convexity.

Really, it is one side of the hypersurface that is pseudoconvex.

If $U \subset \mathbb{C}^n$ is a domain with smooth boundary, $U = \{r < 0\}$, and $dr \neq 0$ near ∂U, then U is *pseudoconvex* if $\partial U = \{r = 0\}$ is pseudoconvex.

If for all $X_p \in T_p^{(1,0)}M$, $X_p = \sum_{k=1}^n a_k \frac{\partial}{\partial z_k}\Big|_p$,

$$X_p^* L_p X_p = \sum_{k=1,\ell=1}^{n} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell}\Big|_p \geq 0,$$

then M is said to be *pseudoconvex* at p.

strictly or *strongly pseucodonvex* if $X_p^* L_p X_p > 0$ for $X_p \neq 0$.

Note the similarity of the definition to classical convexity.

Really, it is one side of the hypersurface that is pseudoconvex.

If $U \subset \mathbb{C}^n$ is a domain with smooth boundary, $U = \{r < 0\}$, and $dr \neq 0$ near ∂U, then U is *pseudoconvex* if $\partial U = \{r = 0\}$ is pseudoconvex.

Pseudoconvex domains are the natural domains of definition for holomorphic functions.

Example 1: $U = \mathbb{B}_2 \subset \mathbb{C}^2$, then $r = |z_1|^2 + |z_2|^2 - 1$.
At $p = (1,0)$, $X_p \in T_p^{(1,0)}\partial U$ means $X_p = a_2 \frac{\partial}{\partial z_2}\Big|_p$ (so $a_1 = 0$)

Example 1: $U = \mathbb{B}_2 \subset \mathbb{C}^2$, then $r = |z_1|^2 + |z_2|^2 - 1$.
At $p = (1,0)$, $X_p \in T_p^{(1,0)}\partial U$ means $X_p = a_2 \frac{\partial}{\partial z_2}\Big|_p$ (so $a_1 = 0$)

The Levi-form (at p) is

$$\sum_{k=1,\ell=1}^{2} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell}\Bigg|_p = \bar{a}_1 a_1 + \bar{a}_2 a_2 = \bar{a}_2 a_2 = |a_2|^2 \geq 0$$

Example 1: $U = \mathbb{B}_2 \subset \mathbb{C}^2$, then $r = |z_1|^2 + |z_2|^2 - 1$.
At $p = (1, 0)$, $X_p \in T_p^{(1,0)} \partial U$ means $X_p = a_2 \frac{\partial}{\partial z_2}\Big|_p$ (so $a_1 = 0$)

The Levi-form (at p) is

$$\sum_{k=1,\ell=1}^{2} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell}\Big|_p = \bar{a}_1 a_1 + \bar{a}_2 a_2 = \bar{a}_2 a_2 = |a_2|^2 \geq 0$$

So $\mathbb{B}_2$ is pseudoconvex (strongly in fact) at p
(and similarly at all $p \in \partial \mathbb{B}_2$).

Example 1: $U = \mathbb{B}_2 \subset \mathbb{C}^2$, then $r = |z_1|^2 + |z_2|^2 - 1$.
At $p = (1, 0)$, $X_p \in T_p^{(1,0)} \partial U$ means $X_p = a_2 \frac{\partial}{\partial z_2}\Big|_p$ (so $a_1 = 0$)

The Levi-form (at p) is

$$\sum_{k=1,\ell=1}^{2} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell}\Bigg|_p = \bar{a}_1 a_1 + \bar{a}_2 a_2 = \bar{a}_2 a_2 = |a_2|^2 \geq 0$$

So $\mathbb{B}_2$ is pseudoconvex (strongly in fact) at p
(and similarly at all $p \in \partial \mathbb{B}_2$).

Example 2: In $(z, w) \in \mathbb{C}^2$, the set the domain $H_+ = \{\operatorname{Im} w > 0\}$ is pseudoconvex at all $p \in M = \partial H_+ = \mathbb{C} \times \mathbb{R}$.

Example 1: $U = \mathbb{B}_2 \subset \mathbb{C}^2$, then $r = |z_1|^2 + |z_2|^2 - 1$.
At $p = (1,0)$, $X_p \in T_p^{(1,0)} \partial U$ means $X_p = a_2 \frac{\partial}{\partial z_2}\Big|_p$ (so $a_1 = 0$)

The Levi-form (at p) is

$$\sum_{k=1,\ell=1}^{2} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell}\Big|_p = \bar{a}_1 a_1 + \bar{a}_2 a_2 = \bar{a}_2 a_2 = |a_2|^2 \geq 0$$

So $\mathbb{B}_2$ is pseudoconvex (strongly in fact) at p
(and similarly at all $p \in \partial \mathbb{B}_2$).

Example 2: In $(z, w) \in \mathbb{C}^2$, the set the domain $H_+ = \{\operatorname{Im} w > 0\}$ is pseudoconvex at all $p \in M = \partial H_+ = \mathbb{C} \times \mathbb{R}$.

But so is the domain $H_- = \{\operatorname{Im} w < 0\}$.

Example 1: $U = \mathbb{B}_2 \subset \mathbb{C}^2$, then $r = |z_1|^2 + |z_2|^2 - 1$.
At $p = (1, 0)$, $X_p \in T_p^{(1,0)} \partial U$ means $X_p = a_2 \frac{\partial}{\partial z_2}\Big|_p$ (so $a_1 = 0$)

The Levi-form (at p) is

$$\sum_{k=1,\ell=1}^{2} \bar{a}_k a_\ell \frac{\partial^2 r}{\partial \bar{z}_k \partial z_\ell}\Big|_p = \bar{a}_1 a_1 + \bar{a}_2 a_2 = \bar{a}_2 a_2 = |a_2|^2 \geq 0$$

So $\mathbb{B}_2$ is pseudoconvex (strongly in fact) at p
(and similarly at all $p \in \partial\mathbb{B}_2$).

Example 2: In $(z, w) \in \mathbb{C}^2$, the set the domain $H_+ = \{\operatorname{Im} w > 0\}$ is pseudoconvex at all $p \in M = \partial H_+ = \mathbb{C} \times \mathbb{R}$.

But so is the domain $H_- = \{\operatorname{Im} w < 0\}$.

So $M = \mathbb{C} \times \mathbb{R} = \{\operatorname{Im} w = 0\}$ is pseudoconvex from both sides
(we call that Levi-flat).

The Levi-form is intrinsic.

The Levi-form is intrinsic.

Let $\pi_p \colon \mathbb{C} \otimes T_p M \to {}^{\mathbb{C} \otimes T_p M}\!\big/_{T_p^{(1,0)} M \oplus T_p^{(0,1)} M} \cong B_p$

be the natural projection.

The Levi-form is intrinsic.

Let $\pi_p \colon \mathbb{C} \otimes T_pM \to {}^{\mathbb{C} \otimes T_pM}\!\big/_{T_p^{(1,0)}M \oplus T_p^{(0,1)}M} \cong B_p$

be the natural projection.

Extend a vector $X_p \in T_p^{(1,0)}M$ to a vector field X in $T^{(1,0)}M$.

The Levi-form is intrinsic.

Let $\pi_p \colon \mathbb{C} \otimes T_pM \to {}^{\mathbb{C} \otimes T_pM}\!\big/\!_{T_p^{(1,0)}M \oplus T_p^{(0,1)}M} \cong B_p$

be the natural projection.

Extend a vector $X_p \in T_p^{(1,0)}M$ to a vector field X in $T^{(1,0)}M$.

Then define the intrinsic Levi-form as

$$\mathscr{L}(X_p, \overline{X}_p) = \pi_p\big([X, \overline{X}]|_p\big)$$

The Levi-form is intrinsic.

Let $\pi_p \colon \mathbb{C} \otimes T_pM \to \mathbb{C} \otimes T_pM \big/ T_p^{(1,0)}M \oplus T_p^{(0,1)}M \cong B_p$

be the natural projection.

Extend a vector $X_p \in T_p^{(1,0)}M$ to a vector field X in $T^{(1,0)}M$.

Then define the intrinsic Levi-form as

$$\mathscr{L}(X_p, \overline{X}_p) = \pi_p\big([X, \overline{X}]|_p\big)$$

This definition works in any codimension, and is completely intrinsic.

The Levi-form is intrinsic.

Let $\pi_p \colon \mathbb{C} \otimes T_pM \to {}^{\mathbb{C} \otimes T_pM}\!\big/_{T_p^{(1,0)}M \oplus T_p^{(0,1)}M} \cong B_p$

be the natural projection.

Extend a vector $X_p \in T_p^{(1,0)}M$ to a vector field X in $T^{(1,0)}M$.

Then define the intrinsic Levi-form as

$$\mathscr{L}(X_p, \overline{X}_p) = \pi_p\big([X, \overline{X}]|_p\big)$$

This definition works in any codimension, and is completely intrinsic.

Exercise: Work out that this definition gives a form that has the same inertia as the previous definition.

Write an M as before as ($T\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is linear)

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w) = q(z, \bar{z}) + (\operatorname{Re} w)(Tz + \overline{Tz}) + a(\operatorname{Re} w)^2 + O(3),$$

Change variables in z to make $Tz = \epsilon z_1$ where $\epsilon = 0$ or 1.

Write an M as before as ($T\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is linear)

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w) = q(z, \bar{z}) + (\operatorname{Re} w)(Tz + \overline{Tz}) + a(\operatorname{Re} w)^2 + O(3),$$

Change variables in z to make $Tz = \epsilon z_1$ where $\epsilon = 0$ or 1.

Change variables changing w to $w + iaw^2 + 2i\epsilon w z_1$ to get

$$\operatorname{Im} w = q(z, \bar{z}) - \epsilon i(\operatorname{Im} w)(z_1 - \bar{z}_1) + a(\operatorname{Im} w)^2 + O(3)$$

Write an M as before as ($T\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is linear)

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w) = q(z, \bar{z}) + (\operatorname{Re} w)(Tz + \overline{Tz}) + a(\operatorname{Re} w)^2 + O(3),$$

Change variables in z to make $Tz = \epsilon z_1$ where $\epsilon = 0$ or 1.

Change variables changing w to $w + iaw^2 + 2i\epsilon w z_1$ to get

$$\operatorname{Im} w = q(z, \bar{z}) - \epsilon i(\operatorname{Im} w)(z_1 - \bar{z}_1) + a(\operatorname{Im} w)^2 + O(3)$$

Solve for $\operatorname{Im} w$ (which is $O(2)$) by IVT to get

$$\operatorname{Im} w = q(z, \bar{z}) + O(3) = z^* A z + z^t B z + \overline{z^t B z} + O(3)$$

Write an M as before as ($T\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is linear)

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w) = q(z, \bar{z}) + (\operatorname{Re} w)(Tz + \overline{Tz}) + a(\operatorname{Re} w)^2 + O(3),$$

Change variables in z to make $Tz = \epsilon z_1$ where $\epsilon = 0$ or 1.

Change variables changing w to $w + iaw^2 + 2i\epsilon w z_1$ to get

$$\operatorname{Im} w = q(z, \bar{z}) - \epsilon i(\operatorname{Im} w)(z_1 - \bar{z}_1) + a(\operatorname{Im} w)^2 + O(3)$$

Solve for $\operatorname{Im} w$ (which is $O(2)$) by IVT to get

$$\operatorname{Im} w = q(z, \bar{z}) + O(3) = z^* A z + z^t B z + \overline{z^t B z} + O(3)$$

Change variables again taking w to $w + 2iz^t Bz$ to get

$$\operatorname{Im} w = z^* A z + O(3)$$

The matrix A is then the Levi-form.

Write an M as before as ($T\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is linear)

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w) = q(z, \bar{z}) + (\operatorname{Re} w)(Tz + \overline{Tz}) + a(\operatorname{Re} w)^2 + O(3),$$

Change variables in z to make $Tz = \epsilon z_1$ where $\epsilon = 0$ or 1.

Change variables changing w to $w + iaw^2 + 2i\epsilon w z_1$ to get

$$\operatorname{Im} w = q(z, \bar{z}) - \epsilon i(\operatorname{Im} w)(z_1 - \bar{z}_1) + a(\operatorname{Im} w)^2 + O(3)$$

Solve for $\operatorname{Im} w$ (which is $O(2)$) by IVT to get

$$\operatorname{Im} w = q(z, \bar{z}) + O(3) = z^* A z + z^t B z + \overline{z^t B z} + O(3)$$

Change variables again taking w to $w + 2iz^t B z$ to get

$$\operatorname{Im} w = z^* A z + O(3)$$

The matrix A is then the Levi-form.

Diagonalizing A and rescaling

$$\operatorname{Im} w = \lambda_1 |z_1|^2 + \cdots + \lambda_{n-1} |z_{n-1}|^2 + O(3) \quad \text{where } \lambda_k = 0 \text{ or } \pm 1.$$

A smooth function $f : M \to \mathbb{C}$ is a *CR function* if

$$vf = 0$$

for all vector fields $v \in \Gamma(T^{(0,1)}M)$.

A smooth function $f : M \to \mathbb{C}$ is a *CR function* if

$$vf = 0$$

for all vector fields $v \in \Gamma(T^{(0,1)}M)$.

Example: If $f \in \mathcal{O}(\mathbb{C}^n)$, then $vf = 0$ for all $v \in \Gamma(T^{(0,1)}\mathbb{C}^n)$. So $f|_M$ is CR.

A smooth function $f : M \to \mathbb{C}$ is a *CR function* if

$$vf = 0$$

for all vector fields $v \in \Gamma(T^{(0,1)}M)$.

Example: If $f \in \mathcal{O}(\mathbb{C}^n)$, then $vf = 0$ for all $v \in \Gamma(T^{(0,1)}\mathbb{C}^n)$. So $f|_M$ is CR.

There are other smooth CR functions.

Example: Suppose $M = \{\operatorname{Im} w = 0\}$, and $f : M \to \mathbb{C}$ is $e^{-(1/\operatorname{Re} w)^2}$ if $\operatorname{Re} w \neq 0$ and 0 if $\operatorname{Re} w = 0$. Then f is CR, C^∞, but f is not real-analytic, so not a restriction of a holomorphic function.

A smooth function $f\colon M \to \mathbb{C}$ is a *CR function* if

$$vf = 0$$

for all vector fields $v \in \Gamma(T^{(0,1)}M)$.

Example: If $f \in \mathcal{O}(\mathbb{C}^n)$, then $vf = 0$ for all $v \in \Gamma(T^{(0,1)}\mathbb{C}^n)$. So $f|_M$ is CR.

There are other smooth CR functions.

Example: Suppose $M = \{\operatorname{Im} w = 0\}$, and $f\colon M \to \mathbb{C}$ is $e^{-(1/\operatorname{Re} w)^2}$ if $\operatorname{Re} w \neq 0$ and 0 if $\operatorname{Re} w = 0$. Then f is CR, C^∞, but f is not real-analytic, so not a restriction of a holomorphic function.

We will see that for real-analytic M and f, CR functions *are* restrictions of holomorphic functions.

A smooth function $f: M \to \mathbb{C}$ is a *CR function* if

$$vf = 0$$

for all vector fields $v \in \Gamma(T^{(0,1)}M)$.

Example: If $f \in \mathcal{O}(\mathbb{C}^n)$, then $vf = 0$ for all $v \in \Gamma(T^{(0,1)}\mathbb{C}^n)$. So $f|_M$ is CR.

There are other smooth CR functions.

Example: Suppose $M = \{\operatorname{Im} w = 0\}$, and $f: M \to \mathbb{C}$ is $e^{-(1/\operatorname{Re} w)^2}$ if $\operatorname{Re} w \neq 0$ and 0 if $\operatorname{Re} w = 0$. Then f is CR, C^∞, but f is not real-analytic, so not a restriction of a holomorphic function.

We will see that for real-analytic M and f, CR functions *are* restrictions of holomorphic functions.

For two CR submanifolds M and N, $f: M \to N$ is a CR mapping if each component of f is a CR function.

A smooth function $f : M \to \mathbb{C}$ is a *CR function* if

$$vf = 0$$

for all vector fields $v \in \Gamma(T^{(0,1)}M)$.

Example: If $f \in \mathcal{O}(\mathbb{C}^n)$, then $vf = 0$ for all $v \in \Gamma(T^{(0,1)}\mathbb{C}^n)$. So $f|_M$ is CR.

There are other smooth CR functions.

Example: Suppose $M = \{\operatorname{Im} w = 0\}$, and $f : M \to \mathbb{C}$ is $e^{-(1/\operatorname{Re} w)^2}$ if $\operatorname{Re} w \neq 0$ and 0 if $\operatorname{Re} w = 0$. Then f is CR, C^∞, but f is not real-analytic, so not a restriction of a holomorphic function.

We will see that for real-analytic M and f, CR functions *are* restrictions of holomorphic functions.

For two CR submanifolds M and N, $f : M \to N$ is a CR mapping if each component of f is a CR function.

M and N are *CR diffeomorphic* if there is a diffeomorphism $f : M \to N$ such that f and f^{-1} are CR.