Singular Levi-flat hypersurfaces (3)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Small review:

For a CR manifold M, a smooth function $f: M \to \mathbb{C}$ is a *CR function* if

$$vf = 0$$

for all vector fields $v \in \Gamma(T^{(0,1)}M)$.

There exist smooth CR functions that are not restrictions of holomorphic functions, but we will show in just a bit that all real-analytic CR functions on a real-analytic CR submanifold are.

Then we will completely locally classify **ALL** real-analytic (nonsingular) Levi-flat hypersurfaces.

Suppose $M \subset \mathbb{C}^n$ is a real-analytic real hypersurface given in $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ as

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w) \quad \text{or} \quad \frac{w - \bar{w}}{2i} = \varphi\left(z, \bar{z}, \frac{w + \bar{w}}{2}\right),$$

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Treat a defining equation as a function of z, $\bar{z}$, w, and $\bar{w}$ independently. Holomorphic implicit function theorem implies a holomorphic Q such that M is given by

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We can also find a basis for the vector fields in $T^{(0,1)}M$:
That is, vector fields in $T^{(0,1)}\mathbb{C}^n$ that vanish on the function $\bar{w} - Q(z, \bar{z}, w)$. The following will work:

$$X_k = \frac{\partial}{\partial \bar{z}_k} + \frac{\partial Q}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{w}}.$$

Theorem (Severi)

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So f is holomorphic in both z and w. □

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• or equivalently, M is “pseudoconvex from both sides”: M divides space near p into two pieces both of which are pseudoconvex at p.

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&= \left(\frac{\partial^2 \bar{Q}}{\partial z_\ell \partial \bar{z}_k} + \frac{\partial Q}{\partial \bar{z}_k} \frac{\partial^2 \bar{Q}}{\partial z_\ell \partial \bar{w}} \right) \frac{\partial}{\partial w} - \left(\frac{\partial^2 Q}{\partial \bar{z}_k \partial z_\ell} + \frac{\partial \bar{Q}}{\partial z_\ell} \frac{\partial^2 Q}{\partial \bar{z}_k \partial w} \right) \frac{\partial}{\partial \bar{w}}
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$\frac{\partial \bar{Q}}{\partial z_\ell}\Big|_0 = 0$, and Levi-flat at the origin implies $\frac{\partial^2 Q}{\partial \bar{z}_k \partial z_\ell}\Big|_0 = \frac{\partial^2 \bar{Q}}{\partial z_k \partial \bar{z}_\ell}\Big|_0 = 0$.

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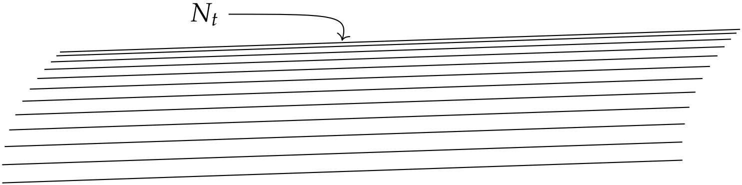

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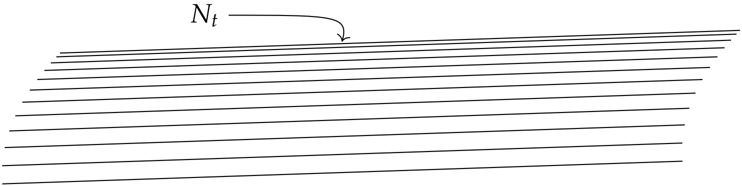

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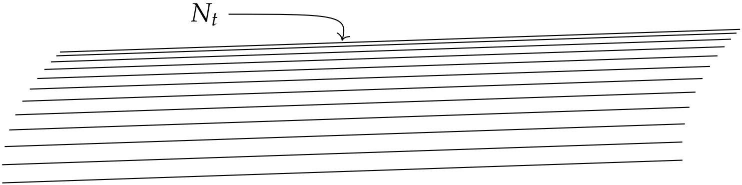

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An example: $M = \{\operatorname{Im} w = 0\}$, here $f = \operatorname{Re} w$.

Theorem (Cartan)

If $M \subset \mathbb{C}^n$ is a Levi-flat real-analytic smooth hypersurface, then near each point $p \in M$, there exist local holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at p such that M near p is given by

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Change variables to make the extended f be the w. □

Exercise: Give the right statement and proof for higher codimension CR manifolds.

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