Singular Levi-flat hypersurfaces (6)

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Small review:

Given a variety $X \subset U \subset \mathbb{C}^n$ defined by $\rho(z, \bar{z})$ converging in $U \times U^*$, the Segre variety is

$$\Sigma_p = \Sigma_p(X, U) = \{z \in U : \rho(z, \bar{p}) = 0\} = \{z \in U : (z, \bar{p}) \in \mathcal{X} = 0\}$$

If X is a real hypersurface in $\mathbb{C}^n$, then Σ_p is usually a complex $(n-1)$-dimensional subvariety.

If Σ_p is n dimensional, X is said to be Segre degenerate.

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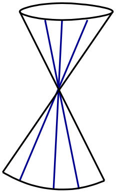
We say X is *Levi-flat* if X^* is Levi-flat at all points.

Example: (Cone)

$X \subset \mathbb{C}^2$ given by

$$\rho(z, w, \bar{z}, \bar{w}) = |z|^2 - |w|^2 = z\bar{z} - w\bar{w} = 0.$$



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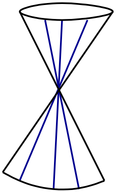

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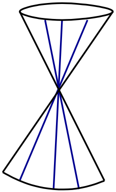

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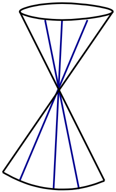

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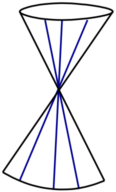

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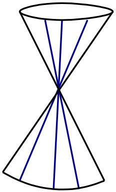

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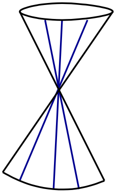

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(If $n = k = 2$, then Σ_0 is the union of lines $z_1 = iz_2$ and $z_1 = -iz_2$)

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Remark: Σ_p is not always guaranteed to be a subset of X for a Levi-flat even if it is not degenerate, just one of its components.

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In other words, all components of $U \setminus \overline{X^*}$ are pseudoconvex.

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As the Levi-form is continuous, X^* is Levi-flat at all points. □

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The singularity is the set where $f_1 = \cdots = f_n = 0$.

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Levi-flat hypersurfaces often arise as invariant sets of holomorphic foliations. However, not all Levi-flat hypersurfaces admit an extension of the Levi-foliation into a holomorphic foliation of a neighborhood (the Brunella '07 example).

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